Math 581 Homework 9

May 9, 2022

Read Chapter 7.

Problem 1 (Problem 7-1).

Problem 2 (Problem 7-2).

Problem 3 (Problem 7-3).

Problem 4 (Problem 7-4).

Problem 5 (Problem 7-5).

Problem 6. Let $S^\infty = \bigcup_{n\geqslant 0} S^n$ be the union of all the spheres, where we regard $S^n \subseteq S^{n+1}$ by adding 0 as the last component of any vector in S^n. Give this the topology where a subset is closed if and only if it is closed when intersected with every S^n.

Prove that S^∞ is contractible. [Hint: Consider the function $\sigma : S^\infty \to S^\infty$ defined by $\sigma(x_1, x_2, ...) = (0, x_1, x_2, ...)$.]

Problem 7. Let X be a set with two unital, binary operations $\circ$ and $\otimes$ and suppose that

$$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$$

for all $a, b, c, d \in X$. Show that $\otimes = \circ$ and both are commutative.

Problem 8. Let (X, x_0) be a based space and let $\Omega X := \mathrm{Map}_*(S^1, X) \subseteq \mathrm{Map}(S^1, X)$ be the subspace of continuous loops based at x_0. Construct a bijection between $\pi_1(X, x_0)$ and the set of path components of ΩX.

Problem 9. Suppose $\pi : E \to X$ is a map equipped with a section $\zeta : X \to E$ and a continuous map $m : \mathbb{R} \times E \to E$ such that: (i) $\pi(m(a, e)) = \pi(e)$ for all $e \in E$, and (ii) $m(0, e) = \zeta(\pi(e))$ for all $e \in E$. Prove that π is a homotopy equivalence. (The prototypical example is if we imagine that each $\pi^{-1}(x)$ is equipped with the structure of a vector space in a manner which varies continuously with x. For example, the projection map $E_{k,n} \to \mathrm{Gr}_k(\mathbb{R}^{n+k})$ from Homework 6.)