Math 581 Homework 8

May 9, 2022

Read Chapter 5, omitting the section on simplicial complexes. Read Chapter 6; you may skip the proof of the classification theorem, but it is good to understand the presentations of the standard compact surfaces.

Problem 1 (Problem 3-13)**.**

Problem 2 (Problem 3-18)**.**

Problem 3 (Problem 4-2)**.**

Problem 4. Prove that if X is a locally finite CW-complex and Y is a CW-complex then $X \times Y$ (with its product topology) is a CW-complex. If X and Y are finite, prove that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

Problem 5. Suppose X is a CW-complex and $A \subseteq X$ is a subcomplex. Show that X/A is a CW-complex. If X is a finite CW-complex, prove that

$$\chi(X) = \chi(X/A) + \chi(A).$$

Problem 6 (Problem 6-3)**.**

Problem 7 (Problem 6-6)**.**

Problem 8. Let X be a CW-complex with n-skeleton denoted $\mathrm{sk}_n X$. Construct a homeomorphism between $\mathrm{sk}_n X/\mathrm{sk}_{n-1} X$ and a wedge of n-spheres (i.e. the quotient of a disjoint union of n-spheres obtained by collapsing all of the basepoints to a single point.)

Problem 9. If J is a finite, linearly ordered set we denote by

$$\Delta^J := \{(x_j)_{j \in J} \in \mathbb{R}^J : \sum x_j = 1, x_j \geqslant 0\ \forall j \in J\}.$$

When $J = [n] := \{0 < 1 < \cdots < n\}$, we give this the shorthand Δ^n and call it the **standard n-simplex**.

(i) Prove that Δ^n is homeomorphic to a closed unit ball of dimension n.

(ii) Prove that Δ^n is homeomorphic to the space

$$\{(t_1, ..., t_n) \in \mathbb{R}^n : 0 \leqslant t_1 \leqslant \cdots \leqslant t_n\}.$$

In other words: we may imagine a point in Δ^n as a configuration of points on the unit interval $[0, 1]$, allowing multiplicities. This visualization can be very useful when making sense of high-dimensional simplices since the image feels '1-dimensional'.

(iii) For each $0 \leqslant i \leqslant n$, define

$$\partial_i \Delta^n := \{(x_0, ..., x_n) \in \Delta^n : x_i = 0\}.$$

(called the **ith face** of Δ^n; it is the one opposite the ith vertex $e_i = (0, 0, ..., 1, ..., 0)$.) And:

$$\partial \Delta^n = \bigcup_{0 \leqslant i \leqslant n} \partial_i \Delta^n.$$

called the **boundary** of the n-simplex. Construct homeomorphisms $\partial_i \Delta^n \simeq \Delta^{n-1}$ and $\partial \Delta^n \cong S^{n-1}$. It may be helpful to understand this construction both in terms of the standard definition of the n-simplex and using the interpretation in the previous part. Also observe that this gives a construction of S^{n-1} as a CW-complex with $n+1$ different $(n-1)$-cells.

(iv) Build a homeomorphism between $\Delta^n \times [0, 1]$ and a union of $n + 1$ copies of Δ^{n+1} glued along faces. (Hint: While this can be 'seen' when $n = 1$ and $n = 2$, in higher dimensions it may be easier to sort out what's happening using (ii).)