Math 581 Homework 7

May 9, 2022

Read the rest of chapter 4.

Problem 1 (Problem 4-34)**.**

Problem 2. Consider the action of $\mathbb{Z}$ on $\mathbb{R}^2 - \{0\}$ given by $n \cdot (x, y) = (2^n x, 2^{-n} y)$.

(a) Show that every point $e \in \mathbb{R}^2 - \{0\}$ has a neighborhood U such that, for every $n \neq 0$, $(n \cdot U) \cap U = \varnothing$. (So it seems like the group action 'moves points to far away points'.)

(b) Nevertheless, show that the quotient $(\mathbb{R}^2 - \{0\})/\mathbb{Z}$ is not Hausdorff (so we are not able to separate orbits by open sets).

Problem 3. We say that the action of a topological group G on a space E is **proper** if the map

$$G \times E \to E \times E, \quad (g, e) \mapsto (ge, e)$$

is proper. Prove that if E is locally compact Hausdorff then E/G is Hausdorff.

Problem 4. If X and Y are topological spaces, define a topology on the set $\mathrm{Map}(X, Y)$ of continuous functions from X to Y with subbasis given by the subsets of the form

$$M(J, U) := \{f : X \to Y | f(J) \subseteq U\}$$

where $J \subseteq X$ is compact and $U \subseteq Y$ is open.

(a) If $g : Y \to Y'$ is a map prove that $g_* : \mathrm{Map}(K, Y) \to \mathrm{Map}(K, Y')$ is continuous where $g_*(f) = g \circ f$.

(b) If K is locally compact Hausdorff and Y is arbitrary, prove that

$$\mathrm{ev} : \mathrm{Map}(K, Y) \times K \to Y$$

given by $\mathrm{ev}(f, x) = f(x)$ is continuous.

(c) If K and Y are arbitrary, prove that the function

$$c : Y \to \mathrm{Map}(K, Y \times K)$$

given by $y \mapsto (k \mapsto (k, y))$ is continuous.

(d) Prove that the following composites are the identity for all Y and K (and they are continuous when K is locally compact Hausdorff):

$$\mathrm{Map}(K, Y) \xrightarrow{\mathrm{Map}(K,c)} \mathrm{Map}(K, \mathrm{Map}(K, Y) \times K) \to \mathrm{Map}(K, Y)$$

$$Y \times K \to K \times \mathrm{Map}(K, Y \times K) \to Y \times K$$

(e) Given $f : Y \times K \to Z$ define $\hat{f} : Y \to \mathrm{Map}(K, Z)$ by $\hat{f}(y)(k) = f(y, k)$. Suppose K is locally compact Hausdorff. Prove that f is continuous if and only if $\hat{f}$ is continuous and that this gives a bijection

$$\mathrm{Hom}_{\mathrm{cts}}(Y \times K, Z) \cong \mathrm{Hom}_{\mathrm{cts}}(Y, \mathrm{Map}(K, Z))$$

where $\mathrm{Hom}_{\mathrm{cts}}(-, -)$ denotes the set of continuous functions (with no topology on it).

(f) Give an alternative proof that, when K is locally compact Hausdorff and $q : X \to Y$ is a quotient map, then $q \times \mathrm{id}_K : X \times K \to Y \times K$ is a quotient map. (Hint: Check that the universal property of the quotient is satisfied by replacing maps out of $Y \times K$ with maps out of Y for some different target.)

Problem 5. Let K be a locally compact, second countable, metric space and let Y be any metric space. Prove that there is a metric on $\mathrm{Map}(K, Y)$ whose associated topology agrees with the one constructed in the previous problem. Prove that we have $f_n \to f$ for a sequence of functions $\{f_n\}$ if and only if the functions f_n converge uniformly to f when restricted to any compact subspace of K. (But they may not converge uniformly on all of K). Recall that $f_n \to f$ converges uniformly on X if for every $\varepsilon > 0$ there is an $N > 0$ so that $d(f_n(x), f(x)) < \varepsilon$ for every $n \geqslant N$ and $x \in X$.

Problem 6. We say that a space Y is **compactly generated** if $U \subseteq Y$ is open if and only if $U \cap K$ is open in K for all $K \subseteq Y$ compact.

(a) Prove that if $q : Y \to Z$ is a quotient map with Y compactly generated, then Z is compactly generated.

(b) Prove that every compactly generated space is the quotient of a locally compact space.

(c) Prove that if K is locally compact Hausdorff, and Y is compactly generated, then $K \times Y$ is compactly generated.