Math 581 Homework 6

October 4, 2022

Read the sections on compactness and local compactness in chapter 4.

Problem 1. Interpret two of the problems from homeworks 1 and 2 as secretly being statements about compactness. Rewrite the statements using the word "compactness".

Problem 2. Use one of the problems in Homework 2 to prove that, if $\{X_\alpha\}_{\alpha \in A}$ is any family of compact spaces, then $\prod X_\alpha$ is compact. (This is called Tychonoff's theorem). Use a homework problem from Homework 4 to give a very strange proof that the unit interval $[0, 1]$ is compact.

Problem 3 (Problem 4-23)**.**

Problem 4 (Problem 4-25)**.**

Problem 5. (a) Let $C \subseteq E$ be a closed subspace of a compact Hausdorff space. Show that E/C is (homeormorphic to) the one-point compactification of $E - C$.

(b) If A and B are spaces with distinguished points a and b respectively, define

$$A \wedge B := (A \times B)/(A \times \{b\} \cup \{a\} \times B)$$

If X and Y are locally compact Huasdorff spaces, prove that

$$(X \times Y)^+ \cong X^+ \wedge Y^+$$

where we use the points at ∞ as the distinguished points. (In particular, if V and W are finite-dimensional vector spaces then

$$(V \oplus W)^+ \cong V^+ \wedge W^+.$$

Some authors write $S^V = V^+$ since V^+ is homeomorphic to a sphere of dimension equal to $\dim_{\mathbb{R}}(V)$, and then we get $S^V \wedge S^W \cong S^{V \oplus W}$. For example, $S^n \wedge S^m \cong S^{n+m}$.)

(c) Let $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ denote the space of k-dimensional (linear) subspaces of $\mathbb{R}^{n+k}$. Here we regard this as a topological space using the quotient topology for the map

$$\mathrm{GL}_{n+k}(\mathbb{R}) \to \mathrm{Gr}_k(\mathbb{R}^{n+k})$$

sending a matrix to the span of its first k columns. Define

$$E_{k,n} = \{(V, x) : x \in V\} \subseteq \mathrm{Gr}_k(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}.$$

Prove that $E_k^+ \cong \mathrm{Gr}_k(\mathbb{R}^{n+k+1})/\mathrm{Gr}_{k-1}(\mathbb{R}^{n+k})$. Here we make sense of the latter quotient using the closed embedding

$$\mathrm{Gr}_{k-1}(\mathbb{R}^{n+k}) \hookrightarrow \mathrm{Gr}_k(\mathbb{R}^{n+k+1})$$

given by $V \mapsto V \oplus \mathbb{R}$. As a special case, show that

$$E_{1,n}^+ \cong \mathbb{R}P^{n+1}$$

(Hint: It may help to think about the graph of the function $\langle x, - \rangle : V \to \mathbb{R}$.)

Problem 6. Without using Tychonoff's theorem, prove directly that a countable product of second-countable, compact spaces is compact. (Hint: Prove that this product is second-countable and then use a criterion for checking compactness on such spaces from the textbook.)