Math 581 Homework 4

April 21, 2022

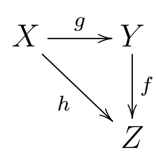
Read the rest of Chapter 3.

Problem 1 (Problem 3-14 and 3-15)**.**

Problem 2 (Problem 3-16)**.**

Problem 3 (Problem 3-17)**.**

Exercise 4. Consider the following commutative diagram of spaces and continuous functions.



(i) Prove that embeddings, quotients, and homeomorphisms are closed under composition.

(ii) Prove that if h is an embedding then so is g.

(iii) Prove that if h is a quotient map then so is f.

(iv) Prove that if any two out of the three functions f, g, and h are homeomorphisms, then so is the third.

Problem 5. Construct a homeomorphism $\mathrm{SO}(3) \cong \mathbb{R}P^3$. (Here $\mathrm{SO}(3)$ denotes the space of 3×3 matrices A such that $\det(A) = 1$ and $A^T A = I$. In other words: the space of rotations in 3-space).

Problem 6. Let $X = \prod_{i=1}^{\infty}\{0, 1\}$ with the product topology (where each $\{0, 1\}$ has the discrete topology).

(i) Show that the function $q : X \to [0, 1]$ given by the formula

$$q((a_i)) = \sum_{i \geqslant 1} a_i 2^{-i}$$

is well-defined (i.e. that the sum converges and lands in the indicated interval), continuous, and surjective.

(ii) Prove that q is a quotient map. (Hint: try proving something stronger.)

Problem 7. We say that a square of spaces

$$\begin{CD} A @>>> B \\ @VVV @VVV \\ A' @>>> B' \end{CD}$$

is a **pushout** if the resulting map

$$A' \cup_A B \to B'$$

is a homeomorphism.

(a) Show that a square as above is a pushout square if and only if it satisfies the following universal property: given any other square

$$\begin{CD} A @>>> B \\ @VVV @VVV \\ A' @>>> C \end{CD}$$

there is a unique map $B' \to C$ making the evident diagram commute. (Part of the exercise is to determine 'the evident diagram').

(b) Suppose we are given a diagram

$$\begin{CD} A @>>> B @>>> C \\ @VVV @VVV @VVV \\ A' @>>> B' @>>> C' \end{CD}$$

and suppose that the leftmost square is a pushout. Prove that the rightmost square is a pushout if and only if the outer rectangle is a pushout.

(c) Given a space X, we say that another space X' is **obtained from X by attaching an n-cell** if we are given $X \to X'$ fitting into a pushout square

$$\begin{CD} \partial\mathbb{D}^n @>>> X \\ @VVV @VVV \\ \mathbb{D}^n @>>> X' \end{CD}$$

where the left vertical arrow is the canonical inclusion. Construct a homeomorphism $X'/X \cong S^n$. (Here $\mathbb{D}^n$ denotes the closed unit disk in $\mathbb{R}^n$ and S^n denotes the n-dimensional sphere, e.g. the unit vectors in $\mathbb{R}^{n+1}$).

(d) Prove that $\mathbb{R}P^{n+1}$ is obtained from $\mathbb{R}P^n$ by attaching an $(n+1)$-cell, and that $\mathbb{C}P^{n+1}$ is obtained from $\mathbb{C}P^n$ by attaching a $(2n+2)$-cell. Deduce that we have homeomorphisms

$$\mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n, \quad \mathbb{C}P^n/\mathbb{C}P^{n-1} \cong S^{2n}.$$

Problem 8 (Problem 3-19)**.**

Problem 9 (Problem 3-20)**.**