Math 581 Homework 3

September 13, 2022

Read Chapter 3 up to, but not including, quotients.

Problem 1 (Problem 3-3)**.**

Problem 2 (Problem 3-5)**.**

Problem 3 (Problem 3-6)**.**

Problem 4 (Problem 3-12, omit (d))**.**

Problem 5. Let $\{(X_i, d_i)\}_{i \geqslant 0}$ be a countable sequence of metric spaces. Use the d_i to construct a metric on $\prod_{i \geqslant 0} X_i$ with underlying topological space that of $\prod_{i \geqslant 0} X_i$ with the product topology. As an example, show that $\{0,1\}^{\times \infty} = \prod_{i=1}^{\infty} \{0,1\}$ is not discrete (given the product topology). [Hint: First prove a lemma that, if (M, d) is a metric space then (M, d') is a metric space where $d'(x,y) := \frac{d(x,y)}{1+d(x,y)}$, but now $d' < 1$ everywhere, and these two metrics give equal topologies. Replace the metrics on X_i by modified ones as above, and try summing them all together. This won't converge... how can you fix it?]

Problem 6. Suppose we are given maps

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & S \end{array}$$

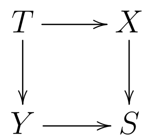
Construct a topological space P and maps $g' : P \to X$, $f' : P \to Y$ such that:

(a) $g \circ f' = f \circ g'$, i.e. the diagram

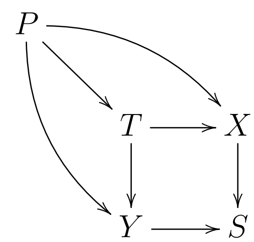
$$\begin{array}{ccc} P & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & S \end{array}$$

commutes.

(b) Given a commutative diagram



there exists a unique map $\phi : P \to T$ such that the larger diagram



commutes.

Prove that the properties (a) and (b) characterize (P, f', g') up to unique homeomorphism compatible with the maps to X and Y. (That is, if (P', f'', g'') also satisfied (a) and (b), then there would be a unique homeomorphism $\phi : P \to P'$ such that $f'' \circ \phi = f'$ and $g'' \circ \phi = g'$).

Problem 7. Let $f : X \to Y$ be a map and define

$$X \times_Y X := \{(a, b) \in X \times X : f(a) = f(b)\} \subseteq X \times X,$$

endowed with the subspace topology. Show that the following are equivalent.

(a) f is a local homeomorphism (i.e. for each $x \in X$ there is an open neighborhood U such that $f(U)$ is open and $f|_U : U \to f(U)$ is a homeomorphism).

(b) f is open and the map $\Delta_f : X \to X \times_Y X$, given by $x \mapsto (x, x)$, is an open embedding.

Problem 8. Given a sequence

$$\cdots \to X_i \to X_{i-1} \to \cdots \to X_1 \to X_0$$

of maps between topological spaces, $s_i : X_i \to X_{i-1}$, define

$$\lim X_i = \{(x_i) : s_i(x_i) = x_{i-1}, i \geqslant 1\} \subseteq \prod_{i \geqslant 0} X_i,$$

equipped with the subspace topology. For a prime p, define

$$\mathbb{Z}_p^\wedge := \lim \left(\cdots \to \mathbb{Z}/p^k \to \mathbb{Z}/p^{k-1} \to \cdots \to \mathbb{Z}/p\right).$$

(a) Show that $\mathbb{Z}_p^\wedge$ is Hausdorff.

(b) Show there is a unique injective, continuous map $\mathbb{Z} \to \mathbb{Z}_p^\wedge$ such that the projection to each $\mathbb{Z}/p^k$ is the canonical one. Prove that this map is *not* a topological embedding.

(c) If $x = p^t n$ is a nonzero integer with n prime to p, write $|x|_p = p^{-t}$; by convention we set $|0|_p = 0$. Suppose that $\{x_i\}$ is a sequence of integers such that $|x_i|_p \to 0$. Prove that $\sum_{i \geqslant 0} x_i$ converges uniquely inside $\mathbb{Z}_p^\wedge$ (i.e. that the sequence of partial sums converges uniquely).

(d) Prove that every element of $\mathbb{Z}_p^\wedge$ can be written in the form

$$\sum_{i \geqslant 0} a_i p^i$$

where $a_i \in \{0, ..., p-1\}$. (Here recall we view integers as elements of $\mathbb{Z}_p^\wedge$ via our embedding $\mathbb{Z} \to \mathbb{Z}_p^\wedge$.)

(e) Show there exists an element $x = (x_i) \in \mathbb{Z}_7^\wedge$ such that $(x_i^2) = 2$. (Hint: try a recursive construction.)