## Math 581 Homework 12

May 9, 2022

Read Chapter 10.

**Problem 1** (10-1).

**Problem 2** (10-2).

Problem 3 (10-3).

Problem 4 (10-4).

**Problem 5** (10-5).

Problem 6 (10-6).

**Problem 7** (10-7).

**Problem 8.** Recall that a **groupoid** is a category all of whose morphisms are invertible. If X is a space we try to define a groupoid  $\pi_{\leq 1}(X)$  with

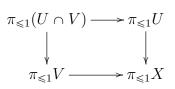
- Objects: the points of x.
- Morphisms: A morphism from x to y is a homotopy class (with fixed endpoints) of path  $\gamma : [0, 1] \to X$  with  $\gamma(0) = x$  and  $\gamma(1) = y$ .
- Composition: If  $\gamma$  is a path from x to y and  $\alpha$  is a path from y to z, the composite is the homotopy class of

$$(\alpha\gamma)(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \alpha(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Prove that:

- (a) Composition is associative.
- (b) Every morphism is invertible.

(c) Prove that if X is a space and  $U, V \subseteq X$  are open with  $X = U \cup V$  then the diagram



is a pushout in the category of groupoids. (Hint: Prove directly that it satisfies the universal property by mimicking the proof of the Van Kampen theorem- but in this case it should actually be *easier* and more natural!)

(d) Let  $A \subseteq X$  be some subset and denote by  $\pi_{\leq 1}(X, A)$  the full subcategory of X whose objects are the points in A. With notation as above, suppose that A contains a point in each component of U, V, and  $U \cap V$ . Prove that the natural inclusion of the square

into the square of the previous part admits a retract.

- (e) Prove that, in any category, a retract of a pushout square is a pushout square.
- (f) Deduce the Van Kampen theorem. (You'll need to find a relationship between pushouts of groupoids and pushouts of groups.)
- (g) Give a new computation of the fundamental group of  $S^1$  using the open cover by  $S^1 \{1\}$ and  $S^1 - \{-1\}$ .