Math 581 Homework 12

May 9, 2022

Read Chapter 10.

Problem 1 (10-1)**.**

Problem 2 (10-2)**.**

Problem 3 (10-3)**.**

Problem 4 (10-4)**.**

Problem 5 (10-5)**.**

Problem 6 (10-6)**.**

Problem 7 (10-7)**.**

Problem 8. Recall that a **groupoid** is a category all of whose morphisms are invertible. If X is a space we try to define a groupoid $\pi_{\leqslant 1}(X)$ with

- Objects: the points of x.
- Morphisms: A morphism from x to y is a homotopy class (with fixed endpoints) of path $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.
- Composition: If γ is a path from x to y and α is a path from y to z, the composite is the homotopy class of

$$(\alpha\gamma)(t) = \begin{cases} \gamma(2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ \alpha(2t-1) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Prove that:

(a) Composition is associative.

(b) Every morphism is invertible.

(c) Prove that if X is a space and $U, V \subseteq X$ are open with $X = U \cup V$ then the diagram

$$\begin{array}{ccc} \pi_{\leqslant 1}(U \cap V) & \longrightarrow & \pi_{\leqslant 1}U \\ \downarrow & & \downarrow \\ \pi_{\leqslant 1}V & \longrightarrow & \pi_{\leqslant 1}X \end{array}$$

is a pushout in the category of groupoids. (Hint: Prove directly that it satisfies the universal property by mimicking the proof of the Van Kampen theorem- but in this case it should actually be *easier* and more natural!)

(d) Let $A \subseteq X$ be some subset and denote by $\pi_{\leqslant 1}(X, A)$ the full subcategory of X whose objects are the points in A. With notation as above, suppose that A contains a point in each component of U, V, and $U \cap V$. Prove that the natural inclusion of the square

$$\begin{array}{ccc} \pi_{\leqslant 1}(U \cap V, A \cap U \cap V) & \longrightarrow & \pi_{\leqslant 1}(U, A \cap U \\ \downarrow & & \downarrow \\ \pi_{\leqslant 1}(V, A \cap V) & \longrightarrow & \pi_{\leqslant 1}(X, A) \end{array}$$

into the square of the previous part admits a retract.

(e) Prove that, in any category, a retract of a pushout square is a pushout square.

(f) Deduce the Van Kampen theorem. (You'll need to find a relationship between pushouts of groupoids and pushouts of groups.)

(g) Give a new computation of the fundamental group of S^1 using the open cover by $S^1 - \{1\}$ and $S^1 - \{-1\}$.