Math 581 Homework 10

November 15, 2022

Read Chapter 9, Appendix C, and the handout on category theory.

Problem 1. Spell out what each of the following categorical concepts work out to be in the special case of posets viewed as categories.

(a) Functors.

(b) Natural transformations.

(c) Equivalence of categories.

(d) Colimits and limits. When do all small (co)limits exist?

(e) Adjoint functors.

Problem 2. Spell out what each of the following categorical concepts work out to be in the special case of groups viewed as categories (via the association of $\mathrm{B}G$ to G).

(a) Functors.

(b) Natural transformations.

(c) Equivalence of categories.

(d) Colimits and limits. Which (co)limits exist and which almost never exist, for nontrivial groups? (The general case is a little more challenging; maybe just think about whether or not there are (co)products, (co)equalizers, and pushout/pullbacks).

(e) Adjoint functors.

Problem 3. Let $[1]$ denote the poset $\{0 < 1\}$ viewed as a category. Prove that the following data are equivalent:

(i) A functor $H : \mathcal{C} \times [1] \to \mathcal{D}$.

(ii) A pair of functors $f, g : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\eta : f \to g$.

Problem 4. Describe concretely what an arbitrary (co)limit looks like in each of the categories Set, Top, and Top_*.

Problem 5. Use the uniqueness of adjoints to prove that $\mathbb{Z}[-]$ is naturally isomorphic to $(\mathrm{Free}_{\mathrm{gp}}(-))_{\mathrm{ab}}$. Then prove just the local version of this statement using the (dual of the) Yoneda lemma, i.e. prove that $\mathbb{Z}[X] \cong (\mathrm{Free}_{\mathrm{gp}}(X))_{\mathrm{ab}}$ by showing that these objects corepresent the same functor on the category of abelian groups.

Problem 6. Let $X : K \to \mathcal{C}$ be a diagram and suppose that K has a *final* object $\infty \in K$. Prove that

$$\operatorname*{colim}_K X = X(\infty).$$

(In particular the colimit exists.)

Problem 7. Prove that the functor π_1 admits neither a left nor right adjoint by exhibiting both a limit and colimit which are not preserved by π_1. (Be sure you are computing limits and colimits in the category of *pointed* spaces.)

Problem 8. Suppose G is a group with the following properties:

(i) There is a pushout diagram

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ G & \longrightarrow & \bullet \end{array}$$

where $\bullet$ denotes the trivial group.

(ii) The map $\mathbb{Z} \to G$ above admits a retract.

Prove that $G_{\mathrm{ab}} \cong \mathbb{Z}$.

Problem 9. Let $X : K \to \mathsf{Grp}$ be a diagram. Suppose we have a description of each group $X(i)$ via generators and relations like

$$X(i) = \langle S_i | R_i \rangle$$

Prove that $\operatorname{colim}_K X$ can be described as

$$\langle \coprod_{i \in K} S_i | R_K \rangle$$

where R_K consists of the disjoint union of all the R_i together with all words of the form $x^{-1}(X(f)(x))$ where $f : i \to j$ is a morphism in K and $x \in S_i$. As a special case, describe pushouts in the category of groups in terms of generators and relations.

Problem 10. (i) Show that

$$c \;\begin{array}{c}\xrightarrow{f}\\[-4pt] \xleftarrow[s]{}\\[-4pt] \xrightarrow[g]{}\end{array}\; d \longrightarrow e$$

is a reflexive coequalizer diagram if and only if

$$c \;\underset{g}{\overset{f}{\rightrightarrows}}\; d \longrightarrow e$$

is a coequalizer diagram.

(ii) Let

$$H \;\begin{array}{c}\xrightarrow{\alpha}\\[-4pt] \xleftarrow[s]{}\\[-4pt] \xrightarrow[\beta]{}\end{array}\; G$$

be a diagram on $\Delta^{\mathrm{op}}_{\leqslant 1}$ in the category of groups. Let R be the equivalence relation generated by $\alpha(h) \sim \beta(h)$ and let N be the smallest normal subgroup of G containing each of the elements $\alpha(h)\beta(h)^{-1}$. Prove that xRy if and only if $xy^{-1} \in N$. (Hint: Put a group structure on G/R and consider the kernel of $G \to G/R$.)

(iii) Conclude that $U : \mathsf{Grp} \to \mathsf{Set}$ preserves reflexive coequalizers.

(iv) Give a counterexample to show that U does not preserve coequalizers in general.