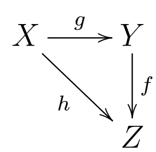
Math 581 Homework 1

April 6, 2022

Read the Introduction, Appendix A, and Appendix B of Lee's *Introduction to Topological Manifolds*.

Exercise 1. Consider the commutative diagram



(i) Prove that if f and g are injective/surjective/bijective then h is also injective/surjective/bijective.

(ii) Prove that if h is injective then so is g.

(iii) Prove that if h is surjective then so is f.

(iv) Prove that if any two out of the three functions f, g, and h are bijective, then so is the third.

(v) Give an example where h is injective but f is not.

(vi) Give an example where h is surjective but g is not.

Problem 2. Let M be a metric space. Prove:

(a) M and $\varnothing$ are open.

(b) Finite intersections of open subsets of M are open.

(c) Arbitrary unions of open subsets of M are open.

Problem 3. Let M be a metric space.

(a) Show that open balls are open and closed balls are closed.

(b) Show that a subset of M is open if and only if it is a union of some collection of open balls.

Problem 4. Prove that the following are equivalent for a metric space M.

(i) Every Cauchy sequence converges.

(ii) Given a sequence of nonempty closed subsets

$$J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$$

with diameters converging to zero, the intersection $\bigcap_{i \geqslant 0} J_i$ is nonempty.

Problem 5. Let M be a metric space. Show that the following are equivalent.

(i) For every collection $\mathfrak{U}$ of open subsets of M such that $\bigcup_{U \in \mathfrak{U}} U = M$, there is a finite subcollection $U_1, ..., U_n \in \mathfrak{U}$ such that

$$M = U_1 \cup \cdots \cup U_n.$$

(ii) If $J_0 \supseteq J_1 \supseteq \cdots$ is a sequence of nonempty closed sets, then $\bigcap_{i \geqslant 0} J_i$ is nonempty.

(iii) Every sequence has a convergent subsequence.

(iv) M is complete and, for every $\varepsilon > 0$, there is a finite collection of open balls of radius ε whose union is M.

Hints: For $(iii) \Rightarrow (iv)$, if there is an ε for which no finite collection of balls suffices, inductively build a sequence of points that are all at least ε far apart from one another. For $(iv) \Rightarrow (i)$, do a proof by contradiction. Prove that for any finite collection of opens covering M, one of the opens must fail to be covered by finitely many of the sets in $\mathfrak{U}$. Apply this observation to the finite covers of M guaranteed by the hypothesis with $\varepsilon = \frac{1}{2^n}$.