Category Primer

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Contents

1 Categories and functors

Definition 1.1. A **category** $\mathcal{C}$ consists of:

- A class[1] of **objects** $\mathrm{ob}(\mathcal{C})$. We write $X \in \mathcal{C}$ to mean $X \in \mathrm{ob}(\mathcal{C})$.
- For each pair $X, Y \in \mathcal{C}$ a set[2] of **morphisms** $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. We write $f : X \to Y$ to indicate that $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a morphism from X to Y.
- for each triple $X, Y, Z \in \mathcal{C}$ a function

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

[1] A category is called **small** if it has a *set* of objects and a set of morphisms, rather than a class.

[2] Technically this is the definition of a 'locally small category'. Occasionally we will consider categories with *classes* of morphisms (can you spot when that happens?), and if one goes further in the subject one has to get concerned about issues of set theory at various points. There are solutions to all these concerns, and I will ignore set theory in this note.

written

$$(f, g) \mapsto f \circ g$$

- For each object X, a distinguished morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$.

This data is subject to the following requirements:

- Composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$ whenever these expressions make sense.

- The element id_X is a unit for composition: $\mathrm{id}_X \circ f = f$ and $g \circ \mathrm{id}_X = g$ whenever these expressions make sense.

Definition 1.2. We say that a morphism $f : X \to Y$ in $\mathcal{C}$ is an **isomorphism** if there is a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In this case we say that g is an inverse for f.

Exercise 1.3. Prove that if f is an isomorphism then it has a *unique* inverse.

Definition 1.4. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of

- an assignment of an object $F(X) \in \mathcal{D}$ for each $X \in \mathcal{C}$.

- an assignment of a morphism $F(f) : F(X) \to F(Y)$ for each morphism $f : X \to Y$ in $\mathcal{C}$.

such that

$$F(f \circ g) = F(f) \circ F(g)$$

whenever this makes sense.

Definition 1.5. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\eta : F \to G$ is an assignment of a morphism $\eta_X : F(X) \to G(X)$ for all $X \in \mathcal{C}$ such that, for every $f : X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

commutes. We say that η is a **natural isomorphism** if there is a natural transformation $\rho : G \to F$ such that $\eta \circ \rho$ and $\rho \circ \eta$ are the identity transformations.

A useful fact is that you can check whether a natural transformation is a natural isomorphism 'pointwise'.

Lemma 1.6. *Let $\eta : F \to G$ be a natural transformation. Then η is a natural isomorphism if and only if $\eta_X : F(X) \to G(X)$ is an isomorphism in $\mathcal{D}$ for all $X \in \mathcal{C}$.*

Proof. One direction is clear. So suppose $\eta_X : F(X) \to G(X)$ is an isomorphism. For each $X \in \mathcal{C}$, let ρ_X be the inverse to η_X. I claim that ρ is natural. If $f : X \to Y$ is a morphism in $\mathcal{C}$ we need to check that $F(f) \circ \rho_X = \rho_Y \circ G(f)$. To prove this, begin with $\eta_Y \circ F(f) = G(f) \circ \eta_X$, then compose on the left with ρ_Y and on the right with ρ_X. □

There is an evident notion of an isomorphism of categories, but more flexible is:

Definition 1.7. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories** if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that FG is naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$ and GF is naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$.

Theorem 1.8. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if the following two conditions hold:*

(i) *F is **fully faithful**: this means that the induced function*

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$$

is a bijection.

(ii) *F is **essentially surjective**: this means that, given any object $d \in \mathcal{D}$ there exists an object $c \in \mathcal{C}$ and an isomorphism $F(c) \cong d$.*

Moreover, if $F : \mathcal{C} \to \mathcal{D}$ satisfies the above two conditions, then the inverse equivalence is unique in the following sense: given $G, G' : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $FG \to \mathrm{id}$ and $FG' \to \mathrm{id}$ that are part of the data of an equivalence, there exists a unique isomorphism $G \cong G'$ compatible with these transformations.

Proof. Let $\mathcal{E}$ be the category whose objects are triples (c, d, α) where $\alpha : F(a) \cong d$ is an isomorphism in $\mathcal{D}$. A morphism is given by morphisms $c \to c'$, $d \to d'$ such that

$$\begin{tikzcd} F(c) \arrow[r] \arrow[d] & d \arrow[d] \\ F(c') \arrow[r] & d' \end{tikzcd}$$

commutes.

The functor F factors as

$$\begin{tikzcd} \mathcal{C} \arrow[r, "j"] \arrow[dr, "F"'] & \mathcal{E} \arrow[d, "p"] \\ & \mathcal{D} \end{tikzcd}$$

where $j(c) = (c, F(c), \mathrm{id}_{F(c)})$ and $p((c, d, \alpha)) = d$. We will show that j and p are equivalences of categories, which implies the claim.

Let $r : \mathcal{E} \to \mathcal{C}$ be defined by $(c, d, \alpha) \mapsto c$. Then $r \circ j = \mathrm{id}_{\mathcal{C}}$. On the other hand, I claim that $j \circ r \cong \mathrm{id}_{\mathcal{D}}$. Indeed, define a natural transformation by

$$\eta_{(c,d,\alpha)} : (c, F(c), \mathrm{id}_{F(c)}) \to (c, d, \alpha)$$

by the morphism in $\mathcal{E}$ specified by the pair $\mathrm{id}_c : c \to c$ and $\alpha : F(c) \to d$. This is a natural isomorphism. (Notice the fact that j was an equivalence did not use any hypotheses about F; it's always true.)

It remains to prove that p is an equivalence. By (ii) we may choose, for each $d \in \mathcal{D}$, some $c \in \mathcal{C}$ and isomorphism $\alpha : F(c) \to d$. This defines a function $s : \mathrm{ob}(\mathcal{D}) \to \mathrm{ob}(\mathcal{E})$. Given a morphism $d \to d'$, the surjectivity part of (i) allows us to choose a morphism $c \to c'$ whose image under F is the composite

$$F(c) \cong d \to d' \cong F(c').$$

This gives an assignment $s : \mathcal{D} \to \mathcal{E}$ on objects and morphisms. That composition is preserved follows from the injectivity assumption in (i).

The uniqueness follows from 'uniqueness of adjoints' proven below. $\square$

2 Limits and colimits

Definition 2.1. Let $X : K \to \mathcal{C}$ be a functor and let $c \in \mathcal{C}$ be an object together with maps $f_i : X(i) \to c$ such that, for every $i \to j$ in K, the diagram

$$\begin{tikzcd} X(i) \arrow[dd] \arrow[dr] & \\ & c \\ X(j) \arrow[ur] & \end{tikzcd}$$

commutes. We say that the morphisms f_i **exhibit** c **as the colimit of** X if, for any other object $d \in \mathcal{C}$ equipped with morphisms $g_i : X(i) \to d$ making the diagrams as above commute, there exists a unique morphism $h : c \to d$ with the property that $h \circ f_i = g_i$ for all $i \in K$. We sometimes summarize this with a picture like so:

$$\begin{tikzcd} X(i) \arrow[dd] \arrow[dr] \arrow[drr, bend left] & & \\ & c \arrow[r, dashed, "!"] & d \\ X(j) \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

Definition 2.2. Let $X : K \to \mathcal{C}$ be a functor and let $c \in \mathcal{C}$ be an object together with maps

$f_i : c \to X(i)$ such that, for every $i \to j$ in K, the diagram

$$\begin{tikzcd} & X(i) \arrow[dd] \\ c \arrow[ur] \arrow[dr] & \\ & X(j) \end{tikzcd}$$

commutes. We say that the morphisms f_i **exhibit c as the limit of X** if, for any other object $d \in \mathcal{C}$ equipped with morphisms $g_i : d \to X(i)$ making the diagrams as above commute, there exists a unique morphism $h : d \to c$ with the property that $f_i \circ h = g_i$ for all $i \in K$. We sometimes summarize this like so:

$$\begin{tikzcd} & & X(i) \arrow[dd] \\ d \arrow[r, dashed, "!"] \arrow[urr, bend left] \arrow[drr, bend right] & c \arrow[ur] \arrow[dr] & \\ & & X(j) \end{tikzcd}$$

Remark 2.3. We like to map **in** to limits and we like to map **out** of colimits.

Remark 2.4. One way to package the data of the maps $X(i) \to c$ above is that we have a natural transformation $X \to \delta(c)$ where $\delta(c) : K \to \mathcal{C}$ is the constant functor at c, i.e. $\delta(c)(i) = c$ and $\delta(c)$ sends every morphism to id_c. Said this way, $X \to \delta(c)$ exhibits c as a colimit if, for every other natural transformation $f : X \to \delta(d)$ there is a unique map $c \to d$ such that $X \to \delta(c) \to \delta(d)$ is f.

Theorem 2.5. *Limits and colimits are unique up to unique isomorphism compatible with structure maps. That is: if $\{f_i : X(i) \to c\}$ and $\{f'_i : X(i) \to c'\}$ exhibit c and c' as colimits of X, then there is a unique isomorphism $h : c \to c'$ such that $h \circ f_i = f'_i$ for all i.*

Proof. By definition we are guaranteed unique maps $h : c \to c'$ and $g : c' \to c$ such that $h \circ f_i = f'_i$ and $g \circ f'_i = f_i$ for all i. The map $g \circ h : c \to c$ has the property that

$$(g \circ h) \circ f_i = g \circ (h \circ f_i) = g \circ f'_i = f_i$$

for all i. But so does the morphism id_c. By the uniqueness clause in the definition of a colimit, we conclude that $g \circ h = \mathrm{id}_c$. Similarly, $h \circ g = \mathrm{id}_{c'}$. Thus h is the sought after isomorphism. $\square$

3 Adjoint functors

Definition 3.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that F is **left adjoint to** G and that G is **right adjoint to** F if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Set}$.

Theorem 3.2 (Uniqueness of adjoints)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $G, G' : \mathcal{D} \to \mathcal{C}$ be functors together with chosen natural isomorphisms*

$$\alpha : \mathrm{Hom}_{\mathcal{D}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

$$\alpha' : \mathrm{Hom}_{\mathcal{D}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G'Y)$$

Then there is a unique natural isomorphism $\beta : G \to G'$ compatible with α and α'.

Proof. The Yoneda lemma (see below) supplies isomorphisms between GY and $G'Y$ for each Y, and one checks that these are natural in Y. □

Theorem 3.3. *Left adjoints preserve colimits and right adjoints preserve limits. More formally: if $X : K \to \mathcal{C}$ is a diagram and $\{f_i : X(i) \to c\}$ exhibits c as a colimit of X, then $\{F(f_i) : FX(i) \to F(c)\}$ exhibits $F(c)$ as a colimit of $F \circ X$.*

Proof. By duality it is enough to prove the claim about colimits.

Suppose we are given $\{g_i : F(X(i)) \to d\}$ making the usual diagrams commute. By adjunction, this is the same data as giving maps

$$\hat{g}_i : X(i) \to G(d)$$

such that, for all $i \to j$, the diagrams

$$\begin{tikzcd} X(i) \arrow[dd] \arrow[dr] & \\ & G(d) \\ X(j) \arrow[ur] & \end{tikzcd}$$

commute. By the definition of a colimit, there is then a unique morphism $\hat{h} : c \to G(d)$ compatible with the structure maps above. This then produces a map $h : F(c) \to d$ by adjunction. This is the desired map. □

4 Examples of categories

Example 4.1. The typical examples consist of 'sets equipped with extra structure, and functions which preserve this structure':

- Set: sets and functions.
- Grp: groups and group homomorphisms.
- Ab: abelian groups and group homomorphisms.
- $\mathsf{Vect}_{\mathbb{R}}$: (real) vector spaces and linear maps.
- Top: topological spaces and continuous maps.
- Top_*: pairs (X, x_0) where X is a space and $x_0 \in X$ is a distinguished point, and continuous maps $f : X \to Y$ such that $f(x_0) = y_0$.
- $h\mathsf{Top}$: topological spaces and homotopy classes of maps.
- $h\mathsf{Top}_*$: pointed spaces and pointed homotopy classes of pointed maps.

Example 4.2. Other examples are often used as 'diagrams', i.e. as the domain of functors we want to take limits or colimits of.

- $\varnothing$: empty.
- Any set S can be regarded as a category with only identity morphisms.
- More generally, any poset P can be regarded as a category with set of objects P and where $\mathrm{Hom}_P(x, y)$ has one object if $x \leqslant y$ and is empty otherwise. (The previous example is the case of posets where every distinct pair of objects is incomparable.) Two special cases come up a lot: one is the poset

  $$0 \longleftarrow 01 \longrightarrow 1$$

  and the other is the poset $\mathbb{Z}_{\geqslant 0}$.
- Any monoid M (i.e. set equipped with a unital, associative binary operation) gives rise to a category with one object $\bullet$ and morphism set M (where $\mathrm{id}_\bullet$ is the unit in M). This category is denoted BM.
- The category Δ whose objects are nonempty, finite linearly ordered sets and whose morphisms are order-preserving functions. We denote by $\Delta_{\leqslant n}$ the category defined the same way but with objects those linearly ordered sets of size at most $n + 1$. (That's unfortunately not a typo... it is the convention to count the 'arrows' in the depiction of a nonempty linearly ordered set rather than the objects.)

Example 4.3. If $\mathcal{C}$ is a category and we are given any subset $S \subseteq \mathrm{ob}(\mathcal{C})$ we can define a new category $\mathcal{D}$ whose objects are those of S and with $\mathrm{Hom}_{\mathcal{D}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$. This is called the **full subcategory spanned by** S. More generally, if we take a sub-collection of objects and sub-collection of morphisms which are closed under composition and contain identity maps, this is called a **subcategory** of $\mathcal{C}$.

Example 4.4. If $\mathcal{C}$ is a category then we can define a new category $\mathcal{C}^{\mathrm{op}}$ with the same objects but with

$$\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(Y,X).$$

Many constructions and theorems come in two flavors and one can pass statements and proofs from one flavor to the other by considering opposite categories. This is called duality.

Example 4.5. If $\{\mathcal{C}_i\}$ is a set of categories then there is an evident notion of $\coprod \mathcal{C}_i$ and $\prod \mathcal{C}_i$ with objects as indicated.

Example 4.6. If $\mathcal{C}$ and $\mathcal{D}$ are categories, we can build a new category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ whose objects are functors and whose morphisms are natural transformations. Notice that, by Lemma 1.6, we can check whether a morphism in $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ is an isomorphism by evaluating the source and target at each element of $\mathcal{C}$. When $\mathcal{C}$ is one of the previous examples, $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ a 'category of diagrams' of some fixed shape in $\mathcal{D}$. The previous example of a product of categories is recovered by taking $\mathcal{C} = S$ a set.

Example 4.7 (Presheaves)**.** A very important special case of the previous example gets singled out with a special name. We define

$$\mathsf{Psh}(\mathcal{C}) := \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}).$$

There is a functor

$$y : \mathcal{C} \to \mathsf{Psh}(\mathcal{C})$$

given by the formula

$$y(c)(x) = \mathrm{Hom}_{\mathcal{C}}(x,c).$$

This functor is called the **Yoneda embedding** (in a moment we will see that it is fully faithful). We regard presheaves as 'generalized objects of $\mathcal{C}$'. When building objects of $\mathcal{C}$ it is often convenient to break up this task into two parts: first build an object in $\mathsf{Psh}(\mathcal{C})$ and then check it lies in the essential image of the Yoneda embedding.

Theorem 4.8 (The Yoneda Lemma)**.** *Let $\mathcal{F} \in \mathsf{Psh}(\mathcal{C})$ be a presheaf and let $c \in \mathcal{C}$ be arbitary. Then the function*

$$\mathrm{Hom}_{\mathsf{Psh}(\mathcal{C})}(y(c), \mathcal{F}) \to \mathcal{F}(c)$$

given by $\eta \mapsto \eta_c(\mathrm{id}_c)$ is a bijection. In particular, y is fully faithful.

Definition 4.9. We say that a presheaf $\mathcal{F} \in \mathsf{Psh}(\mathcal{C})$ is **representable** if it lies in the essential image of y. If $\mathcal{F} \cong y(c)$ we say that c **represents** $\mathcal{F}$.

Corollary 4.10. *Let* $\mathsf{Psh}^{\mathrm{rep}}(\mathcal{C}) \subseteq \mathsf{Psh}(\mathcal{C})$ *be the full subcategory spanned by the representable functors. Then the Yoneda embedding factors through an equivalence* $\mathcal{C} \simeq \mathsf{Psh}^{\mathrm{rep}}(\mathcal{C})$.

Example 4.11. Let $X : K \to \mathcal{C}$ be a diagram, and consider the presheaf $\widetilde{\lim}_K X$

$$c \mapsto \mathrm{Hom}_{\mathsf{Fun}(K,\mathcal{C})}(\delta(c), X)$$

where $\delta : \mathcal{C} \to \mathsf{Fun}(K, \mathcal{C})$ sends an object to the corresponding constant diagram. Then X has a limit if and only if $\widetilde{\lim}_K X$ is representable (in which case any choice of limit provides a representing object).

Example 4.12. A functor $F : \mathcal{C} \to \mathcal{D}$ admits a right adjoint if and only if the functor

$$\mathcal{D} \to \mathsf{Psh}(\mathcal{C})$$

given by $d \mapsto (c \mapsto \mathrm{Hom}_{\mathcal{C}}(Fc, d))$ factors through the full subcategory of representable presheaves. This observation implies that a right adjoint to F is essentially unique if it exists (in the same sense that the inverse to an equivalence of categories is unique if it exists).

Example 4.13. The functor

$$\mathrm{Open}(-) : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$$

sending a topological space to its set of open subsets (with functoriality given by taking preimages) is representable. What is the representing object?

5 Examples of functors

Example 5.1. $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ is a functor, and it factors through $h\mathsf{Top}_*$.

Example 5.2. All of the prototypical first examples of categories (Example 4.1) admit a functor

$$U : \mathcal{C} \to \mathsf{Set}$$

which 'forgets' any added structure. The letter U stands for 'underlying'; this functor is usually called a **forgetful functor** -though sometimes that terminology is reserved only for the algebraic examples. With the exception of Top and Top_*, the forgetful functor has the property that it is **conservative** which means that $f : X \to Y$ is an isomorphism if and only if $U(f)$ is an isomorphism. This is one of the ways that 'algebra' behaves differently than 'geometry'.

Example 5.3. If L and P are posets regarded as categories then a functor $f : L \to P$ is precisely an order-preserving function.

Example 5.4. Any group homomorphism $\phi : G \to H$ gives rise to a functor $\mathrm{B}G \to \mathrm{B}H$ by applying ϕ to the morphisms.

We defined (co)limits only up to unique isomorphism, but in practice we behave as if 'taking a limit' is a well-defined construction. This is justified by the following theorem (which of course has a dual statement for colimits).

Theorem 5.5. *Let* $\mathsf{Fun}'(K, \mathcal{C}) \subseteq \mathsf{Fun}(K, \mathcal{C})$ *denote the full subcategory of diagrams whose limit exists. Then there is a functor*

$$\lim_K : \mathsf{Fun}'(K, \mathcal{C}) \to \mathcal{C}$$

equipped with a natural transformation $\varepsilon : \delta \circ \lim \to \mathrm{id}$ *such that, for every* $X \in \mathsf{Fun}'(K, \mathcal{C})$*, the maps*

$$\{\varepsilon_j : \lim_K X \to X(j)\}$$

exhibit the source as a limit of X*. When* $\mathsf{Fun}'(K, \mathcal{C}) = \mathsf{Fun}(K, \mathcal{C})$*, the functor* $\lim_K$ *is right adjoint to the diagonal* δ*.*

Proof. First define a functor

$$\mathsf{Fun}(K, \mathcal{C}) \to \mathsf{Psh}(\mathcal{C})$$

by

$$X \mapsto (c \mapsto \mathrm{Hom}_{\mathsf{Fun}(K,\mathcal{C})}(\delta(c), X)).$$

Now conclude by examining the diagram:

$$\begin{tikzcd}
\mathsf{Fun}'(K, \mathcal{C}) \arrow[r] \arrow[d] \arrow[rr, bend left, dashed] & \mathsf{Psh}^{\mathrm{rep}}(\mathcal{C}) \arrow[d] & \mathcal{C} \arrow[l, "\simeq"'] \\
\mathsf{Fun}(K, \mathcal{C}) \arrow[r] & \mathsf{Psh}(\mathcal{C}) &
\end{tikzcd}$$

□

6 Examples of (co)limits

Example 6.1. If $K = \varnothing$ there is only one diagram of shape K. A colimit over the empty set in $\mathcal{C}$ is called an **initial object**, and a limit over the empty set in $\mathcal{C}$ is called a **final object**. For example:

(i) The initial object in Set is $\varnothing$. Any singleton set is a final object in Set.

(ii) The initial object in Grp is the trivial group $\{e\}$; this is also the final object. The same is true in Ab and $\mathsf{Vect}_{\mathbb{R}}$.

(iii) If P is a poset then an element is initial if and only if it is an absolute minimum and final if and only if it is an absolute maximum.

Example 6.2. If $K = S$ is a set, then a diagram $X : S \to \mathcal{C}$ is just the data of a collection of objects $\{X_s\}_{s\in S}$. A colimit for this diagram is called a **coproduct** and denoted

$$\coprod_{s\in S} X_s.$$

Similarly, a limit is called a **product** and denoted $\prod_{s\in S} X_s$. For example:

(i) Coproducts in Set are disjoint unions. Products are cartesian products.

(ii) Products in Grp and Ab are usual products. Coproducts in Grp are somewhat complicated and sometimes called 'amalgamated products', denoted $G * H$. Coproducts in Ab are direct sums (so finite coproducts coincide with products), and similarly in $\mathsf{Vect}_{\mathbb{R}}$.

(iii) If P is a poset then a coproduct of some subset of P is a least upper bound, while a product of some set of elements is a greatest lower bound.

Example 6.3. A diagram indexed on $\bullet \to \bullet$ is just a map $X \to Y$ in $\mathcal{C}$. The colimit of such a diagram is Y and the limit is X. More generally, if K admits a final object 1 then a colimit for $F : K \to \mathcal{C}$ always exists and is given by $F(1)$. Dually, if K admits an initial object 0 then then a limit for $F : K \to \mathcal{C}$ always exists and is given by $F(0)$. (Be careful about the possibly confusing mismatch of *initial* objects of K- a colimit- with the *limit* of a diagram indexed on K, etc.)

Example 6.4. If M is a monoid, then a diagram $X : \mathrm{BM} \to \mathcal{C}$ is the same information as an object $X(\bullet) \in \mathcal{C}$ together with a map of monoids $M \to \mathrm{Hom}_{\mathcal{C}}(X, X)$; i.e. an action of M on X. We denote colimits and limits for this diagram by

$$X_M, \text{ and } X^M$$

respectively, and call them **orbits** and **fixed points** for the action of M. Sometimes the first of these is denoted X/M instead.

Example 6.5. When $K = \Delta_{\leqslant 1}$ then a functor $X : \Delta_{\leqslant 1}^{\mathrm{op}} \to \mathcal{C}$ is equivalent to the data of three maps

$$\begin{tikzcd} R \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y \arrow[l, "s_0" description] \end{tikzcd}$$

satisfying the identities $d_0 s_0 = d_1 s_0 = \mathrm{id}_Y$. A colimit for such a diagram is called a **reflexive coequalizer**, sometimes denoted

$$\mathrm{coeq}\left(\begin{tikzcd} R \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y \arrow[l, "s_0" description] \end{tikzcd}\right).$$

It is a generalization of a 'quotient by an equivalence relation.' Dually, we can consider diagrams $X : \Delta_{\leqslant 1} \to \mathcal{C}$ whose limits are called **reflexive equalizers**.

Actually, the map s_0 turns out not to be so relevant here: you get an equivalent (co)limit if you consider just the 'fork'.

Example 6.6. If K is the poset

$$0 \leftarrow 01 \rightarrow 1$$

then a diagram $X : K \to \mathcal{C}$ looks like

$$\begin{tikzcd} X_{01} \arrow[r] \arrow[d] & X_1 \\ X_0 & \end{tikzcd}$$

and a colimit is called a **pushout** or sometimes the **cobase change of** $X_{01} \to X_0$ **along** $X_0 \to X_1$. Dually, given a diagram $X : K^{\mathrm{op}} \to \mathcal{C}$ a limit is called a **pullback** or sometimes a base change.

Example 6.7. If K corresponds to the poset $\mathbb{Z}_{\geqslant 0}$ then a diagram $X : \mathbb{Z}_{\geqslant 0} \to \mathcal{C}$ is just a sequence of maps

$$X_0 \to X_1 \to \cdots$$

and a colimit is called a **sequential colimit**. (Confusingly, in older literature, this is refereed to as an 'inductive limit' or 'direct limit', both of which use the word 'limit' even though this is a colimit.) Similarly, we have limits for diagrams $X : \mathbb{Z}_{\geqslant 0}^{\mathrm{op}} \to \mathcal{C}$ which are **sequential limits** (and again in older literature have funny names like 'projective limit'.)

Incidentally, all colimits can be built out of coproducts and coequalizers if they exist.

Theorem 6.8. *A category $\mathcal{C}$ has all (small) colimits if and only if it has reflexive coequalizers and all (small) coproducts. In this case, if $X : K \to \mathcal{C}$ is a diagram then*

$$\operatorname*{colim}_K X = \mathrm{coeq}\left(\coprod_{f:i\to j} X_i \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}} \coprod_k X_k \right).$$

Dually:

Theorem 6.9. *A category $\mathcal{C}$ has all (small) limits if and only if it has reflexive equalizers and all (small) products. In this case, if $X : K \to \mathcal{C}$ is a diagram then*

$$\lim_K X = \mathrm{eq}\left(\prod_k X_k \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}} \prod_{f:i\to j} X_j \right).$$

Another important example of a colimit is the 'canonical presentation' of presheaves by representable presheaves.

Theorem 6.10. *Let $\mathcal{C}$ be a small category and $X \in \mathsf{Psh}(\mathcal{C})$ a presheaf. Denote by $\mathcal{C} \downarrow X$ the category whose objects are pairs (C, x) where $C \in \mathcal{C}$ and $x \in X(C)$, and where a morphism $(C, x) \to (C', x')$ consists of a map $f : C \to C'$ such that $X(f)(x') = x$. Consider the diagram:*

$$\mathcal{C} \downarrow X \to \mathcal{C} \xrightarrow{y} \mathsf{Psh}(\mathcal{C})$$

Then

$$X = \operatorname*{colim}_{(C,x)\in\mathcal{C}\downarrow X} y(C).$$

7 Examples of adjoint functors

Example 7.1 (Free and forget)**.** The functor $\mathrm{Free}_{\mathrm{gp}} : \mathsf{Set} \to \mathsf{Grp}$, sending a set to the free group on that set, is left adjoint to the forgetful functor $U : \mathsf{Grp} \to \mathsf{Set}$. That is: the datum of a group homomorphism $FX \to G$ is the same as the datum of a function between sets $X \to G$.

Similarly, there is a functor $\mathbb{Z}[-] : \mathsf{Set} \to \mathsf{Ab}$ which assigns, to each set, the free **abelian** group on that set. The groups $\mathbb{Z}[X]$ and $\mathrm{Free}_{\mathrm{gp}}(X)$ almost never coincide, the exceptions being $X = \varnothing$ and $|X| = 1$.

It is also worth pointing out that the forgetful functors in each case do *not* admit right adjoints because these forgetful functors do not preserve colimits. Actually, they preserve some colimits but not others: the forgetful functors here preserve reflexive coequalizers but do not preserve coproducts. Similarly, each of the free functors do not preserve limits and hence do not admit left adjoints.

Example 7.2 (Abelianization)**.** The embedding $\mathsf{Ab} \to \mathsf{Grp}$ admits a left adjoint (but not a right adjoint) denoted $(-)_{\mathrm{ab}} : \mathsf{Grp} \to \mathsf{Ab}$ and computed by the formula:

$$G_{\mathrm{ab}} = G/[G,G], \quad [G,G] = \{ghg^{-1}h^{-1} : g, h, \in G\}.$$

Unwinding the definitions, we learn that the map $G \to G_{\mathrm{ab}}$ is initial amongst all homomorphisms out of G with target an abelian group. More precisely: given any homomorphism $\phi : G \to A$ where A is abelian, there exists a unique homomorphism $G_{\mathrm{ab}} \to A$ such that the composite $G \to G_{\mathrm{ab}} \to A$ is ϕ (i.e. a unique factorization of ϕ through the projection $G \to G_{\mathrm{ab}}$).

Example 7.3 (Posets)**.** If $f : L \to M$ is an order-preserving function between posets, regarded as a functor, then we can ask: when is it the case that f admits a left or right adjoint? In order for f to be a left adjoint, then for every $y \in M$ we must have an element $g(y) \in L$ with the property that $f(x) \leqslant y$ if and only if $x \leqslant g(y)$. In other words, we are forced to define:

$$g(y) = \max(x : f(x) \leqslant y).$$

So f admits a right adjoint if and only if each of these maxima actually exist (notice also that each of these is an example of a colimit). Similarly, f admits a left adjoint if and only if each of the minima

$$\min(x : y \leqslant f(x))$$

exist (in which case this would be the value of the left adjoint at x).

Example 7.4 (Geometric realization)**.** If J is a finite, linearly ordered set then let $\Delta^J \subseteq \mathbb{R}^J$ denote the subset of those functions $x : J \to \mathbb{R}$ with nonnegative values which sum to 1. When $J = [n] := [0 < 1 < \cdots < n]$ we denote this by Δ^n. For every order-preserving function $f : J \to J'$ we can define a continuous function

$$f_* : \Delta^J \to \Delta^{J'}$$

by $f_*((x_j))_\ell = \sum_{j \in f^{-1}(\ell)} x_j$, where we interpret the empty sum as 0. This gives a functor

$$\Delta^\bullet : \Delta \to \mathsf{Top}$$

Now we define a category $\mathsf{sSet} := \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})$ called the category of simplicial sets, and a functor

$$\mathrm{Sing}_\bullet : \mathsf{Top} \longrightarrow \mathsf{sSet}$$

defined as the composite

$$\mathsf{Top} \xrightarrow{y} \mathsf{Fun}(\mathsf{Top}^{\mathrm{op}}, \mathsf{Set}) \to \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{sSet}) = \mathsf{sSet}.$$

Explicitly:

$$\mathrm{Sing}_\bullet(Y) = \mathrm{Hom}_{\mathsf{Top}}(\Delta^J, Y).$$

This functor is a right adjoint, and its left adjoint is denoted by

$$|-| : \mathsf{sSet} \longrightarrow \mathsf{Top}$$

and called **geometric realization**. An explicit formula for it is as follows:

$$|X_\bullet| = \left(\coprod_{f : J \to J'} \Delta^J \times X_{J'} \right) / (f_* a, b) \sim (a, f^* b).$$