CHAPTER 9

Induction

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§1. Process and method of induction

(An Introduction for Teachers). Almost everyone has once had fun arranging dominoes in a row and starting a wave. Push the first domino and it topples the second, the second will topple the third and so forth until all the dominoes are toppled. Now let us change the set of dominoes into an infinite series of propositions: P_1, P_2, P_3, numbered by positive integers. Assume that we can prove that:

(B): the first proposition of the series is true;

(S): the truth of every proposition in the series implies the truth of the next one.

Then, in fact, we have already proved all the propositions in the series. Indeed, we can "push the first domino", i.e., prove the first statement (B), and then statement (S) means that each domino, in falling, topples the next one. Whatever the "domino" (proposition) we choose, it will be eventually hit by this wave of "falling dominos" (proofs).

This is a description of the method of mathematical induction (MMI). Theorem (B) is called *base of induction*, and theorem (S) is *the inductive step.* Our reasoning with the wave of falling dominoes shows that step (S) is but a shortened form of the chain of theorems shown in the figure below:

$$P_1 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow \qquad \longrightarrow P_k \longrightarrow P_{k+1} \longrightarrow$$

We will call theorems in this chain "steps", and the process of their successive proof—"the process of induction" This process can be visually represented as a wave of proofs, running from statement to statement along a chain of theorems.

Psychologically, the essence of induction is in its process. How can we teach this? We will try to show you in a dialog between teacher ("T") and student ("S"), which roughly resembles a session of a real mathematical circle. At the end of the dialog some methodological comments for the teacher are given (references to these comments are indicated in the text of the dialog).

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Problem 1. T: One box was cut off from a 16×16 square of graph paper. Prove that the figure obtained can be dissected into trominos of a certain type—"corners" (see Figure 64.)

S: But this is easy—any "corner" has three boxes, and $16^2 - 1$ is divisible by 3.

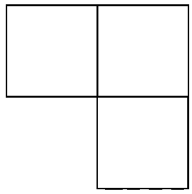

FIGURE 64

T: If it is so easy, could you cut a 1×6 band into "corners"? Six is also divisible by 3!

S: Well Actually, I should not have said that. I don't exactly know how to solve this problem.(1)

T: OK, you cannot solve *this* problem. Perhaps you can think of another problem which is similar yet easier?

S: Well, you can take another square, of smaller size, say, 4×4.

T: Or 2×2?(2)

S: But there is nothing to prove in this case—when you cut out any box what you get is just a "corner" What sense does that make?

T: Try now to solve the problem about the 4×4 square.

S: Uh-huh. A 4×4 square can be cut into four 2×2 squares. It is clear what to do with the one with the cut box. What about the other three?

T: Try to cut a "corner" from them, located in the center of the big square (see Figure 65).

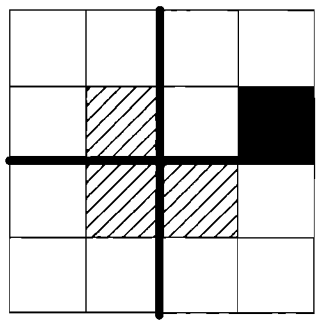

FIGURE 65

S: Got it! Each of them would lack one box and turn into a "corner" So we can solve the problem for a 4×4 square too. Now?

T: Try an 8×8 square. It can be dissected into four 4×4 squares. Make use of this.

S: Well, we can reason as we did before. One of those squares has the "missing" box in it. And we have already proved that this one can be cut into "corners" The three other squares will lack one box after we cut out one "corner" in the center of 8×8 square—so we will be able to dissect them, too.

T: Do you see now how to solve the original problem?

S: Sure. We cut the 16×16 square into four 8×8 squares. One of them contains the cut box. We have just proved that it can be dissected into "corners", right? Then we cut out a "corner" in the center of the big square and we get three more 8×8 squares, each without one box, and each can be cut into "corners" That's it!

T: Not yet. We solved this problem by building "bridges" from similar but simpler questions. Could we build such bridges once more, to other, more complicated questions?(3)

S: Of course. Let us prove that one can dissect a 32×32 square into "corners" We just divide it mentally into four 16×16 squares

T: There you are! But is it possible to go further?

S: Certainly. Having proved the proposition for a 32×32 square we can now derive, in the very same way, a method of dissection for a 64×64 square, then for a 128×128 square and so on

T: Thus, we have an infinite chain of propositions about squares of different sizes. Can we say that we have proved them all?

S: Yes, we have. First, we proved the first statement in the chain—about a 2×2 square. Then we derived the second proposition from it, then the third from the second, et cetera. It seems quite clear that

going along this chain we will reach any of its statements; therefore, all of them are true.

T: Right. It looks like a "wave of proofs" running along the chain of theorems: $2 \times 2 \longrightarrow 4 \times 4 \longrightarrow 8 \times 8 \longrightarrow$ It is quite evident that the wave will not miss any statement in this chain.

* * *

Methodological remark. A few comments on the previous dialog.

Comment $\mathcal{N}^{\underline{\circ}}$ 1. When the student "proved" the statement of the problem using divisibility by 3, the teacher faced a typical classroom problem—how to explain the nature of the error, without giving away too much. The teacher overcomes this with a counterexample, prepared beforehand. It is always useful to be aware of such obstacles and know some ways to avoid them. This must be done easily, without major distraction from the flow of solution.

Comment $\mathcal{N}^{\underline{\circ}}$ 2. This retort is not accidental. The student can hardly think about the 2×2 case as something important—it's not a problem at all (we will come across this psychological moment several times). However, the teacher knows this case is easier to start with.

Comment $\mathcal{N}^{\underline{\circ}}$ 3. The following "step-by-step" scheme appears in this part of the dialog:

$$2 \times 2 \longrightarrow 4 \times 4 \longrightarrow 8 \times 8 \longrightarrow 16 \times 16$$

We have here the beginning of the induction process: the base 2×2 and the first three steps. It is essential that we have made enough induction steps for the student to notice an analogy. Now, after the hint, he is able to develop the whole process of induction.

In fact, there are other inductive solutions to this question but they would not yield any educational benefit, since the notion of induction in them is not as clear as in the solution given above. Thus, the teacher leads the student away from these, using directive hints. The teacher here has played his part precisely: sometimes he leads away from a deceptive analogy and helps to save the student's energy. Unobtrusiveness is very important: the more the student does on his/her own the better.

Let us sum up the results. The student (but more often this is the responsibility of the teacher) explained the scheme of MMI. The underlined sentence ("going along this chain we will reach any of its statements") is but an informal statement of the principle of mathematical induction which is the cornerstone of MMI. You can read about the formal side of it in any of the books [**76, 78, 79**]. We must say, though, that it would not be wise to talk about this at the very beginning of the discussion. It may be premature or even harmful since formalization of this intuitively clear statement may give rise to feelings of misunderstanding and uncertainty. On the contrary, one must use all means to make this scheme as evident and vivid as possible. Aside from the "wave" and dominoes (see Figure 66), other useful analogies include climbing a staircase, zipping a zipper, et cetera.

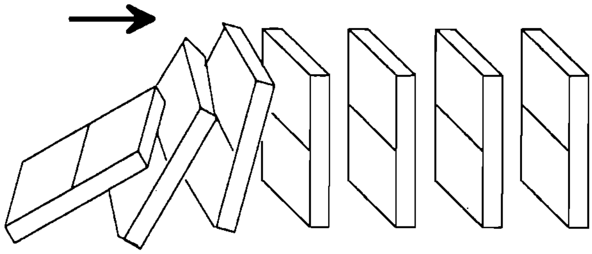

FIGURE 66

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Now let us go on with our dialog:

T: So, we have proved an infinite series of statements about the possibility of dissecting squares into "corners" Now, we write them all down, without any "et cetera's"

S: But we will certainly run out of paper.

T: Yes, we would, if we wrote each statement separately. But all the statements look alike. Only the size of squares differ. This fact allows us to encode the whole chain in just one line:

(*) A $2^n \times 2^n$ square with one box cut out can be dissected into "corners".

Here we have the variable n. Each statement in our chain can be obtained by replacing n with a number. For instance, $n = 5$ gives us a proposition about the 32×32 square. And what is the tenth proposition in the chain?

S: We substitute $n = 10$ to get the statement about $2^{10}\times 2^{10}$, i.e., the 1024×1024 square.

T: Look at this: a variable is such a common thing, but it is really powerful—it allows us to fold an infinite chain into one short sentence. So, what is "a variable"?

S: Well it is just a letter an unknown

T: Remember: this "letter" denotes an empty space, a room, where you can put various numbers or objects. You can also call it a "placeholder" Those numbers or objects that are allowed to be put into the "room" are called its possible values. For example, the values of the variable n in (*) are the natural numbers (positive integers). Because of this, sentence (*) replaces the infinite chain of statements.

Now we must recall the proof of the infinite chain (*). Let us number all the statements: P_1 is the one about the 2×2 square, P_2 is about the 4×4 square, and so on.

First we proved proposition P_1. Then we dealt with the infinite chain of similar theorems: if P_1 is proved, then P_2 is true; if P_2 is proved, then P_3 is true, et cetera. Let us try to encode this chain also: "For any natural n

S: if P_n is true, then P_{n+1} is also true."

T: And now, please, decode this phrase: what do P_n and P_{n+1} denote?

S:

(**) "Whichever natural number n is, if it is already proved that the $2^n \times 2^n$ square without one box can be cut into "corners", then it is also true that the $2^{n+1} \times 2^{n+1}$ square without one box can be cut into "corners"."

T: Can you prove that?

S: I think so. We mentally divide the $2^{n+1} \times 2^{n+1}$ square into four $2^n \times 2^n$ squares. One of these lacks one box, and can be dissected into "corners" by assumption. Then we cut out one "corner" in the center of the big square so that it contains one box from each of the other three $2^n \times 2^n$ squares. After that, we can use the assumption again!

T: Absolutely. Note that as soon as you proved the general theorem (**), you proved all the theorems from the chain encoded by (**). For example, if $n = 1$, we get our old proof stating that the possibility of dissecting the 2×2 square implies the possibility of dissecting the 4×4 square. Therefore, just as (**) can be considered as encoding a whole chain of theorems, your reasoning can be considered as encoding a whole "wave of proofs" of those theorems. I believe you got it: it is useful and easier to prove a chain of similar theorems in this convoluted way. But first you must learn how to express a chain of theorems this way.

* * *

The method we applied in solving Problem 1 is what we call the METHOD OF MATHEMATICAL INDUCTION (MMI). What is its essence?

First, we regard statement (*) not as one whole fact but as an infinite series of similar propositions.

Second, we prove the first proposition in the series—this is called the "base of the induction."

Third, we derive the second proposition from the first, the third (in the same way) from the second, et cetera. That was the "inductive step"; (**)—is its shortened (convoluted) form. Since, step by step, we can reach any proposition from the base, they are all true.

"A method is an idea applied twice"
(G. Polya)

To learn MMI successfully it is usually necessary to replay the scenario above for several different questions. Consider now four more "key problems"

Problem 2. Prove that number $111\ldots11$ (243 ones) is divisible by 243.

Hint. This question may be generalized to the proposition that a number written with 3^n ones is divisible by 3^n

Base: 111 is divisible by 3. Students often start with the statement that 111,111,111 is divisible by 9—our base sounds too easy to them.

Here we have two obstacles

a) an attempt to generalize the divisibility tests for 3 and 9 and use an incorrect "test" for divisibility by 27;

b) reasoning of the sort: "if a number is divisible by 3 and 9, then it is divisible by $27 = 3 \times 9$."

The correct kind of inductive step is to divide the number written with 3^{n+1} ones by the number written with 3^n ones and check that the result is a multiple of 3.

Problem 3. Prove that for any natural number n, greater than 3, there exists a convex n-gon with exactly 3 acute angles.

Comment. This question is a good key problem if students already know the fact that a convex polygon cannot have more than 3 acute angles. The base $n = 4$ can be checked by direct construction.

Inductive step: let us "saw off" one of the non-acute angles. Then the number of angles in the polygon increases by 1 and all the acute angles are retained (see Figure 67).

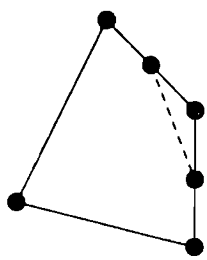

FIGURE 67

Another way to do this—to build a new angle on one of the sides—is a bit more difficult. There are also other solutions (using inscribed polygons and so on) but most are more difficult for students to make precise. Perhaps the teacher can even give a hint about "sawing off" an angle.

The statement of the question is obviously true for $n = 3$, but we will not gain anything by starting the induction from 3, because the method fails when you try to make the step from $n = 3$ to $n = 4$.

Our third question gives an example of construction by induction. You can read about it in more detail in [**79**].

Problem 4. ("Tower of Hanoi") Peter has a children's game. It has three spindles on a base, with n rings on one of them. The rings are arranged in order of their size (see Figure 68). It is permitted to move the highest (smallest) ring on any spindle onto another spindle, except that you cannot put a larger ring on top of a smaller one. Prove that

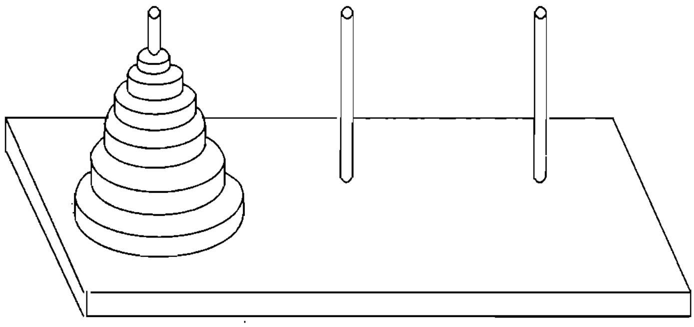

FIGURE 68

a) It is possible to move all the rings to one of the free spindles;
b) Peter can do so using $2^n - 1$ moves.
c)*It is not possible to do so using fewer moves.

Hint. a), b): The base (n=1) is easy.

Inductive step: We have $n = k + 1$ rings. By the inductive assumption we can move all but the largest ring to the third spindle using $2^k - 1$ moves. Then we move the remaining ring to the second spindle. After that we can move all the rings from the third spindle to the second using $2^k - 1$ moves. In all, we have made $(2^k - 1) + 1 + (2^k - 1) = 2^{k+1} - 1$ moves. It is useful to do the first few steps of the induction "manually", even using a physical model.

c) This question must be used with care—it is more difficult than the others given here. The main idea of the proof is that to move the widest ring to the second spindle, we must first move all the other rings to the third spindle.

Problem 5. The plane is divided into regions by several straight lines. Prove that one can color these regions using two colors so that any two adjacent regions have different colors (we call two regions *adjacent* if they share at least one line segment).

Hint. Here we encounter another obstacle: no explicit variable for induction is given in the statement. Thus, we should start the solution by revealing this hidden variable. To do this, we can rewrite the statement as follows: "There are n straight lines on a plane   The base can be $n = 1$ or $n = 2$ (either will work). The inductive step: remove for a moment the $(k + 1)$st line, color the map obtained, then restore the removed line and reverse the colors of all the regions on one side of the line.

<u>**For teachers.**</u> The first few key problems can be discussed according to the scenario of the dialog above; that is, growing the chain from one particular proposition. Students should understand the essence of the process of induction and the connection between chains of theorems and propositions using integer variables.

If students are not well prepared one can skip the idea of constructing a chain of inductive steps. This can be introduced later, at a second stage, whose goal is to teach the students to work with the inductive step in its convoluted form. While doing so it would be wise to give questions in a general form (like in Problems 3 and 4). There we already have a chain of statements and the solution may start right from the "unfolding", as follows: "Here we have a convoluted chain of theorems. What is the first theorem? The fifth? The 1995th?" However, the chain of inductive steps should be developed and convoluted according to the old scheme, until students get accustomed to it and understand well the connection between a long chain and its convoluted form.

* * *

To sum up their experience with key problems, students should have a clear

General Plan for Solution by the Method of Mathematical Induction

1. Find, in the statement of the question, a series of similar propositions. If variables are hidden you should reveal them by reformulating the question. If there is no chain, try to grow that chain so that the question will be a part of it.
2. Prove the first proposition (base of the induction).
3. Prove that for any natural number n the truth of the nth proposition implies the truth of the $(n+1)$st proposition (inductive step).
4. If the base and the step are proved, then all the propositions in the series are proved simultaneously, since you can reach any of them from the base by moving "step-by-step"

The last item in this scheme is the same for all the problems, so it is often skipped. However, knowing it is vital. Also, the first item is not emphasized and is natural for those who are used to MMI; nevertheless we recommend that the students pay close attention to it for a while.

§2. MMI and guessing by analogy

We continue our dialog.

Problem 6. Into how many parts do n straight lines divide a plane if no two of them are parallel and no three meet at the same point? (Figure 69 shows an example where $n = 5$.)

S: Let us try to follow the scheme. Do we have a chain? It seems so: into how many parts does one line divide a plane? 2 lines? 3 lines ?

The base is evident: one line dissects a plane into 2 parts (half-planes).

T: Or zero lines—into one part.

S: By all means. Item three—the inductive step !?

T: I can understand your embarrassment: we run into a new difficulty. In the previous problems we dealt with chains of statements, not with chains of questions. But we will get a chain of statements if we give hypothetical, unproved answers to these questions.

S: How can we?

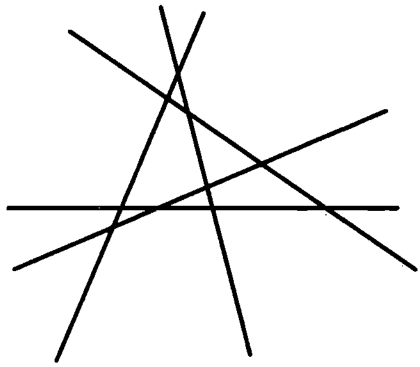

FIGURE 69

T: Try to guess a rule, a function giving the number of parts L_n in terms of the number of lines n. Physicists would do an experiment. We can experiment too, calculating the numbers L_n for small values of n. Go ahead!

S: OK. So, $L_0 = 1$, $L_1 = 2$, $L_2 = 4$, $L_3 = 7$, $L_4 = 11$. I must think a little .Ah, I got it! When you add the nth line the number of parts increases by n. Hence, $L_n = 1 + (1 + 2 + \quad + n)$. I did it!

T: No, not yet. Don't forget that you have only guessed it, not proved it. You have checked your result only for $n = 0, 1, 2, 3, 4$. For all other values of n this is just a guess based on your conjecture that adding the nth line increases the number of parts by n. What if this is wrong? The only guarantee is a proof.

S: by the method of mathematical induction.

T: But we should enhance our plan from §1 by another item:

1a. If there is a chain of questions rather than a chain of statements in a mathematical problem, insert your hypothetical answers. You can guess the answers by experimenting with the first few questions in the chain. However, after you are sure the answers are correct, don't forget to prove them rigorously.

S: Now I know how to get over this. We have already proved the base, right? To prove the inductive step is easy: the nth line intersects the other lines at $n - 1$ points, which divide the line into n parts. Therefore the nth line divides n of the old parts of the plane into new parts.

* * *

The process of guessing by analogy, just demonstrated by our student, is a very powerful and, sometimes, very dangerous tool: it is tempting to mistake the regularity one finds as a proof. The two examples below can serve as good medicine for this disease.

Problem 7. Is it true that the number n^2+n+41 is prime for any natural number n?

Hint. The answer is no: For $n = 40$ we have $40^2 + 40 + 41 = 41^2$, and for $n = 41$ $41^2 + 41 + 41 = 41 \cdot 43$. But anyone trying to find an answer by "experimenting" with small values of n would come to the opposite conclusion, since this formula gives prime numbers for n from 1 through 39. This famous example was given by Leonard Euler.

Problem 8*. A set of n points is taken on a circle and each pair is connected by a segment. It happens that no three of these segments meet at the same point. Into how many parts do they divide the interior of the circle?

Hint. For $n = 1, 2, 3, 4, 5$ we obtain 1, 2, 4, 8, and 16 respectively. This result provokes a guess to the formula 2^{n-1} However, in fact, the number of parts equals $\frac{n(n-1)(n-2)(n-3)}{24} + \frac{n(n-1)}{2} + 1$.

Other similar examples can be found in [**78**].

§3. Classical elementary problems

Among classical MMI problems in elementary mathematics three large groups can be distinguished: proofs of identities, proofs of inequalities and proofs of divisibility questions. Though their solutions by MMI seem to be quite simple, students usually encounter some obstacles of a psychological as well as of a methodical nature. We begin by discussing these.

T: Let us talk more about Problem 6. Do you like the way the formula $1 + (1 + 2 + 3 + \quad + n)$ looks?

S: Not much. It is too bulky. It would be better to get rid of this ellipsis (the three dots).

T: No problem. You can prove by MMI that $1 + 2 + 3 + \quad + n = n(n+1)/2$.

S: But to use MMI you need a chain of statements

T: Take a close look: there is variable n in the formula. As we know, this is a good sign of a convoluted set of problems. Substitute, for instance, 1995 for n.

S: We get $1 + 2 + \quad + 1995 = 1995 \quad 1996/2$.

T: That is, a numerical equation. Our convoluted set of problems consists of all these equations (for $n = 1, 2, 3, \quad , 1995$)! To prove the formula means to show that all these numerical equations are true. If we do this, we say that this equation is "true for all admissible values of the variable" and it is called an *identity*. If an identity contains an integer variable you can try to prove it by induction.

S: What if our equation is not true for some n?

T: Then it is not an identity and we will not be able to prove it—the proof of either the base or the inductive step will not go through. Actually, to distinguish between identities and other, arbitrary equations with variables, you must preface identities with phrases like "for any natural number n it is true that ", but this is not the usual practice. It is implied that the reader knows from the context whether an identity or a conditional equation is being discussed.

S: Well, let us apply MMI. Base: $n = 1$. So we must prove that $1 + 2 + \quad + 1 = 1 \quad 2/2 = 1$?!

T: No, no. We must prove that $1 = 1 \quad 2/2$. You were puzzled by the formula $1+2+\ldots+n$. This is quite good and convenient, but for $n = 1$ its "tail" $2+\ldots+n$ makes no sense and, in fact, does not exist at all.

S: OK, so the base is clear. Let us move to the second equation in the series. We must show that $1 + 2 = 2 \quad 3/2$. This is easy: $3 = 3$. Now, move to the third equation: $1 + 2 + 3 = 3 \cdot 4/2$. This is easy too: $6 = 6$. To the fourth it is just another simple calculation. So, what now? Must we check each equation directly? We haven't got any steps!

T: Try to rewrite the step in a general, convoluted form.

S(after a while): I cannot do that either.

For teachers. To a person who has mastered MMI well enough, the proof of identities may seem rather trivial. However, our dialog shows two sources of problems for students. First, students often do not accept an identity with an integer variable as a chain of statements. This is probably because simple numerical identities are not considered independent propositions. Also, what is interesting in a statement such as $1+2+3=3\quad 4/2=6$?

Second, it is next to impossible to see how the general form of the inductive step looks. Indeed, when you check the equations $1+2=2\cdot 3/2$, $1+2+3=3\cdot 4/2$, and so on, there is no connection between two successive facts—you just check them.

That is why identities, despite their simplicity, cannot serve as key questions. To start learning and teaching mathematical induction from these will create trouble (this is not very important for really gifted students—they will manage to learn the method in any case). On the other hand, identities are very useful for practice, because their proofs are usually short and clear.

T: Well, I will help you. Imagine that we follow the steps of the induction, one after another and the wave of proofs have reached the kth statement. What is that statement?

S: We obtain $1+2+3+\quad +k=k(k+1)/2$. (#)

T: Exactly. Now, tell me, please, what is the next statement, which the wave has not yet reached?

S: Certainly, $n=k+1$ and we get $1+2+\quad+(k+1)=(k+1)(k+2)/2$.

T: Good. Let us write this as follows:

$$1+2+3+\quad +k+(k+1)=\frac{1}{2}(k+1)(k+2). \qquad (\#\#)$$

Now, tell me what would be the next step of induction?

S: That's clear: to derive (##) from (#).

T: Assume that we learned how to derive (##) from (#) for any natural number k. Then we would have all the steps of induction proved at once. This means that the inductive step states that:

For any natural k the equation $1+2+\ldots+k=k(k+1)/2$ implies the equation $1+2+\quad+(k+1)=(k+1)(k+2)/2$.

In other words: (#) is given, and we must prove (##) (if k is an arbitrary natural number). For convenience we denote the left sides of (#) and (##) as S_k and S_{k+1} respectively.

S: Proposition (##) shows that $S_{k+1}=S_k+(k+1)$ (that is why the teacher has written the next-to-last summand!). Now we have already learned that $S_k=k(k+1)/2$. Thus we have

$$S_{k+1}=\frac{1}{2}k(k+1)+(k+1)=\frac{1}{2}\big[k(k+1)+2(k+1)\big]=\frac{1}{2}(k+1)(k+2).$$

T: Remember the helpful idea that we used to prove the inductive step: the left side of equation (##) was expressed with the left side of (#) and the latter was substituted into the right side of (#).

For teachers. Another difficulty now arises in connection with identities. It may not be clear to a student how to make a step "in letters" The teacher in our dialog showed how to overcome that. It is important that he used another letter—different from that used in the statement of the identity—to denote the variable. The point is that the letter k plays the role not of a variable but of a constant (though arbitrary) number marking the place that the wave of our inductive proof reached at the given moment. It will become a variable later, in the general statement of the inductive step.

Quite often, the variables in the statement of the proposition and in the step are both denoted by the same letter. While stating the step theorem, phrases like " now we substitute $n+1$ in place of n" are used. This is not advisable in the beginning of the study since it disorients most students conceptually (it is hard to see a chain in the statement of the inductive step) as well as technically (it is not that easy for a beginner to substitute $n+1$ for n).

Now we can say goodbye to the characters in our dialog and go on to deal with problems. Problems 9–16 are about identities with the natural number n as their variable.

Problem 9. Show that $1+3+ \quad +(2n-1)=n^2$

Problem 10. Show that $1^2+2^2+ \cdot\cdot+n^2=n(n+1)(2n+1)/6$.

Problem 11. Show that $1 \quad 2+2 \quad 3+\cdots+(n-1) \quad n=(n-1)n(n+1)/3$.

Problem 12. Show that

$$\frac{1}{1 \quad 2}+\frac{1}{2\cdot 3}+ \quad +\frac{1}{(n-1)n}=\frac{n-1}{n}$$

Problem 13. Show that

$$1+x+x^2+\cdot \quad +x^n=(x^{n+1}-1)/(x-1)$$

Problem 14. Show that

$$\frac{1}{a(a+b)}+\frac{1}{(a+b)(a+2b)}+ \quad \cdot+\frac{1}{(a+(n-1)b)(a+nb)}=\frac{n}{a(a+nb)}$$

where a and b are any natural numbers.

Problem 15. Show that

$$\frac{m!}{0!}+\frac{(m+1)!}{1!}+ \quad +\frac{(m+n)!}{n!}=\frac{(m+n+1)!}{n!(m+1)},$$

where m, $n=0, 1, 2,$

Problem 16. Show that

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right) \quad \left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

Comments. In Problems 9–15 the proof of the inductive step is exactly the same as in the dialog. However, in Problem 16, it may be proved more easily by representing the $(k+1)$st left side not as a sum but as the product of the kth left side and $(1-\frac{1}{k^2})$. This trick may also be useful in proving certain inequalities (see below).

In Problem 11 the base is not $n=1$ but $n=2$. Students should see that this doesn't influence the process of induction.

In Problem 15 induction is possible on either of the two variables. It is instructive to carry out and compare both proofs. Remember to start from zero!

Problems 11 and 12 are special cases of Problems 15 (for $m = 2$) and 14 (for $a = b = 1$) respectively. Given other values of m, a, and b we obtain any number of exercises like 11 and 12. It would be wise to let good students try to find the statement of the general problem which generates these exercises.

Most of the identities 9–16 have good non-inductive proofs which are not too difficult. Problem 9 has a neat geometric proof (see Figure 70). Identity 11 can be obtained from identities 9 and 10. Identity 13 can be proved by division of $x^{n+1} - 1$ by $x - 1$, and identity 16 by direct calculation. To prove identity 12 it suffices to note that its left side equals

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

and that this sum "telescopes"

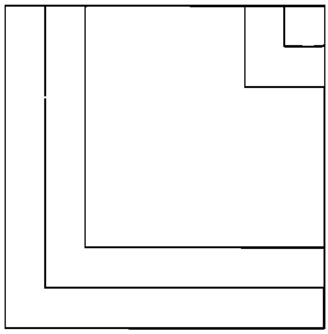

FIGURE 70

This device works for other identities too.

Discussion of these alternative proofs can be very helpful to students who have already mastered MMI.

Divisibility questions constitute the next natural step in our study. The techniques of forming statements and inductive steps are similar to those for identities: we usually find the increment of the expression under consideration and prove that it is divisible by a given number. Problems 17–19 have simple alternative solutions (using modular arithmetic). The rather difficult Problem 22 may serve as the source of a number of exercises like 18–19.

Prove that for any natural number n

Problem 17. $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Problem 18. $3^{2n+2} + 8n - 9$ is divisible by 16.

Problem 19. $4^n + 15n - 1$ is divisible by 9.

Problem 20. $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Problem 21. $2^{3^n} + 1$ is divisible by 3^{n+1}.

Problem 22: $ab^n + cn + d$ is divisible by the positive integer m given that $a + d$, $(b-1)c$, and $ab - a + c$ are divisible by m.

Our trio of standard MMI themes is completed by questions involving inequalities. Here the proofs of the inductive steps are usually more varied (see [78]). Prove the following inequalities:

Problem 23. $2^n > n$, where n is an arbitrary natural number.

Problem 24. Find all natural numbers n such that

a) $2^n > 2n + 1$; b) $2^n > n^2$

Problem 25.

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{13}{24}, \quad n = 2, 3, \ldots$$

Problem 26. $2^n > 1 + n\sqrt{2^{n-1}}$, $n = 2$, 3,

Problem 27. Prove that the absolute value of the sum of several numbers does not exceed the sum of the absolute values of these numbers.

Problem 28. $(1+x)^n > 1 + nx$, where $x > -1$, $x \neq 0$, and $n = 2$, 3,

Problem 29.

$$\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n} \leq \frac{1}{\sqrt{2n+1}},$$

where n is any natural number.

Hints. 23, 24: To prove the inductive step you may show that for any n, the increment of the left side of the inequality is greater than the increment of the right side.

24b: Use 24a to prove the step.

25: Prove that the left side of the inequality is monotonically increasing.

27: Induction can proceed on the number of summands.

28, 29: See the hint to Problem 16.

§4. Other models of MMI

So far we have been dealing with the basic version of MMI. When this is well learned we can try other, more complicated forms of induction. Some of these can be considered corollaries of the basic form, but it is more natural from a methodological point of view to discuss them separately, keeping in mind the image of "a wave of proofs"

First, consider the method "Induction from all $n \leq k$ to $n = k+1$", sometimes called "strong induction"

In the usual method of MMI, the inductive step consists of deriving proposition P_{k+1} from the previous proposition P_k. Sometimes, however, to show the truth of P_{k+1} we must use more than one (or even all) of the previous statements P_1 through P_k. This is certainly valid, since the wave has reached P_k and, therefore, all the propositions in the chain preceding it are also already proved. Thus the statement of the inductive step is:

(S'): For any natural k the truth of P_1, P_2, . . and P_k implies the truth of P_{k+1}.

Consider an example.

Problem 30. Prove that every natural number can be represented as a sum of several distinct powers of 2.

Solution. First, let us prove the base. If the number given equals 1 or 2, then the existence of the required representation is simple.

Now denote the given number by n and find the largest power of 2 not exceeding n. Let it be 2^m; that is, $2^m \leq n < 2^{m+1}$ The difference $d = n - 2^m$ is less than n and also less than 2^m, since $2^{m+1} = 2^m + 2^m$ By the induction hypothesis, d can be represented as a sum of several different powers of 2, and it is clear that 2^m is too big to be included. Thus, adding 2^m we get the required expression for n. The induction is complete.

Problem 31. Prove that any polygon (not necessarily convex) can be dissected into triangles by disjoint diagonals (they are allowed to meet only at vertices of the polygon).

Hint. Use an induction on the number of sides. The inductive step is based on a lemma stating that each polygon (except a triangle) has at least one diagonal which lies completely within the polygon. Such a diagonal dissects the polygon into two polygons with fewer sides.

* * *

Another scheme of MMI is demonstrated by

Problem 32. It is known that $x + 1/x$ is an integer. Prove that $x^n + 1/x^n$ is also an integer (for any natural n).

Solution. We have $(x^k + 1/x^k)(x + 1/x) = x^{k-1} + 1/x^{k-1} + x^{k+1} + 1/x^{k+1}$ and hence $x^{k+1} + 1/x^{k+1} = (x^k + 1/x^k)(x + 1/x) - (x^{k-1} + 1/x^{k-1})$. So we see that the $(k + 1)$st sum is an integer if the two preceding sums are integers. Thus the process of induction will go as usual if we check that the first two sums, $x + 1/x$ and $x^2 + 1/x^2$, are integers. This is left to the reader.

Comment. A peculiarity of this version of MMI is that the inductive step is based on two preceding propositions, not one. Thus, the base in this case consists of the first two propositions in the series (it is natural to use the word *base* for that starting segment of the chain in which the statements must be checked directly).

Problem 33. The sequence a_1, a_2, , a_n, of numbers is such that $a_1 = 3$, $a_2 = 5$, and $a_{n+1} = 3a_n - 2a_{n-1}$ for $n > 2$. Prove that $a_n = 2^n + 1$ for all natural numbers n.

Hint. See the more general Problem 43.

Remark. In Problem 33 and the next three problems we will encounter not only proof by induction but also definitions by induction: all elements of the given sequences, except for the first few, are defined by induction, using the preceding elements. Sequences defined in this way are called *recursive*; see [**75**] and [**77**] for more details. See also [**79**], Chapter 2, about definitions, constructions, and calculations using induction.

Problem 34. The sequence (a_n) is such that: $a_1 = 1$, $a_2 = 2$, $a_{n+1} = a_n - a_{n-1}$ if $n > 2$. Prove that $a_{n+6} = a_n$ for all natural numbers n.

Problem 35. The sequence of Fibonacci numbers is defined by: $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ if $n \geq 2$. Prove that any natural number can be represented as the sum of several different Fibonacci numbers.

Problem 36*. Prove that the nth Fibonacci number is divisible by 3 if and only if n is divisible by 4.

Hint. It is not easy to prove this fact alone by induction. Prove a more general statement about the repetition of remainders of Fibonacci numbers modulo 3 (with period 8). If you want to know more about Fibonacci numbers, see [**75**].

Problem 37. A bank has an unlimited supply of 3-peso and 5-peso notes. Prove that it can pay any number of pesos greater than 7.

Hint. Try induction on the number of pesos the bank must pay. The base consists of three facts: $8 = 5 + 3$, $9 = 3 + 3 + 3$, $10 = 5 + 5$. Inductive step: if the bank can pay k, $k + 1$, and $k + 2$ pesos, then it can pay $k + 3$, $k + 4$, and $k + 5$ pesos. This induction with a compound base may be split into three standard inductions using the following schemes:

$$8 \to 11 \to 14 \to \qquad 9 \to 12 \to 15 \to \ldots, \text{ and } 10 \to 13 \to 16 \to$$

Note that a similar splitting is impossible in Problems 32–36.

There also exists a non-inductive solution to this problem based on the equations $3n + 1 = 5 + 5 + 3(n - 3)$ and $3n + 2 = 5 + 3(n - 1)$, but it is not easier than the solution above.

The following three questions are very close to Problem 37.

Problem 38. It is allowed to tear a piece of paper into 4 or 6 smaller pieces. Prove that following this rule you can tear a sheet of paper into any number of pieces greater than 8.

Problem 39. Prove that a square can be dissected into n squares for $n \geq 6$.

Problem 40. Prove that an equilateral triangle can be dissected into n equilateral triangles for $n \geq 6$.

Comments. 38: If we tear a piece of paper into 4 or 6 smaller pieces, then the number of pieces increases by 3 or 5 respectively. Now we use the method of solution from Problem 37.

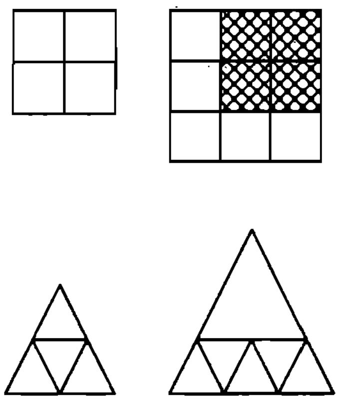

FIGURE 71

39, 40: A square (equilateral triangle) can be dissected into 4 or 6 squares (equilateral triangles) as shown in Figure 71. Thus Problem 39 can be reduced to Problem 38. There exist other non-inductive solutions based on the possibility of cutting a square (equilateral triangle) into any even number (greater than 2) of squares (or equilateral triangles) greater than 2—see Figure 72.

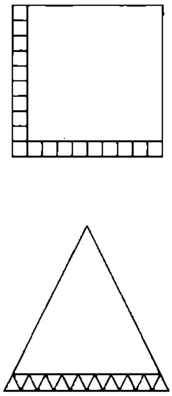

FIGURE 72

Other schemes of induction are even more exotic. An example is the method of "ramifying induction" which enables us to give a proof of a remarkable inequality for the arithmetic and geometric means.

Problem 41*. Prove that for any non-negative numbers x_1, x_2, , x_n

$$\frac{x_1 + x_2 + \quad + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$

Sketch of proof. The base: $n = 2$ is rather simple. Then you must use steps from $n = 2^k$ to 2^{k+1} in order to prove the inequality for all n equal to power of 2. And finally, you prove that if the inequality is true for any n numbers, then it is true for any $n - 1$ numbers. The wave of proofs spreads in accordance with the scheme in Figure 73.

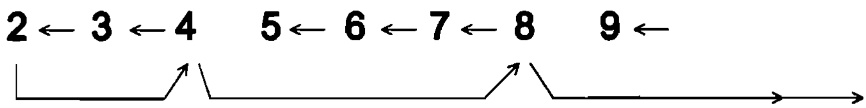

FIGURE 73

See details in [**78**] (example 24), and also in the chapter "Inequalities"

Schemes involving "backwards induction" (over negative integers) and "double (or, 2-dimensional) induction" (for theorems involving two natural parameters) are illustrated in Problems 43 and 44.

§5. Problems with no comments

Problem 42. Two relatively prime natural numbers m, n, and the number 0 are given. A calculator can execute only one operation: to calculate the arithmetic mean of two given natural numbers if they are both even or both odd. Prove that using this calculator you can obtain all the natural numbers 1 through n, if you can enter into the calculator only the three numbers initially given or results of previous calculations.

Problem 43. For the sequence $a_1, a_2, \ldots$ from Problem 33 we can define elements $a_0, a_{-1}, a_{-2}, \ldots$ so that the equation $a_{n+1} = 3a_n - 2a_{n-1}$ will hold true for any integer n (positive or negative). Prove that the equality $a_n = 2^n + 1$ will still be true for all integers n.

Problem 44. Prove that $2^{m+n-2} \geq mn$ if m and n are positive integers.

Problem 45*. Several squares are given. Prove that it is possible to cut them into pieces and arrange them to form a single large square.

Problem 46*. Prove that among any 2^{n+1} natural numbers there are 2^n numbers whose sum is divisible by 2^n

Problem 47. What is the greatest number of parts into which n circles can dissect a plane? What about n triangles?

Remark. Compare Problem 6. Examples of the required dissections can also be done by induction.

Problem 48. Several circles are drawn on a plane. A chord then is drawn in each of them. Prove that this "map" can be colored using three colors so that the colors of any two adjacent regions are different.

* * *

Problem 49*. Prove by "reductio ad absurdum" that the principle of mathematical induction stated in the very beginning of the present chapter is equivalent to the following "well order principle": in any non-empty set of natural numbers there exists a least element. Try to rewrite the solution of one of the previous problems (say, Problem 46) using this principle and compare it to the proof by induction.

For more about the well order principle and its applications, see [**19**], pp.88–96.

* * *

Conclusion. The method of mathematical induction is a very helpful and useful idea. You will find its applications in various places in this book, as well as in other mathematical contexts. However, we would like to warn you against an "addiction" to it. You should not think that any question with statements and/or proofs using the words "et cetera" or "similarly" is a problem to be solved by MMI. Proofs by induction for many of those questions (you will see some of them in the chapters "Graphs–2" and "Inequalities") look rather artificial compared to other proofs involving such simple methods as direct calculation or recursive reasoning. It is not advisable to use such unnatural examples when teaching the nature of MMI, although they can be used well after the method is completely mastered.