Math 303 Homework 9

Proofs by induction

Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Prove by induction that there is a real number a such that $f(n) = an$ for all $n \in \mathbb{N}$.

(b) Deduce that $f(n) = an$ for all $n \in \mathbb{Z}$.

(c) Deduce further that $f(x) = ax$ for all $x \in \mathbb{Q}$.

Hint: To find the value of a in part (a), try substituting some small values of x and y into the equation $f(x + y) = f(x) + f(y)$. For part (b), use the fact that $n + (-n) = 0$ for all $n \in \mathbb{N}$. For part (c), consider the number $bf(\frac{a}{b})$ when $a, b \in \mathbb{Z}$ and $b \neq 0$.

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(0) > 0$ and $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Prove that there is some positive real number a such that $f(x) = a^x$ for all *rational* numbers x.. Hint: Like in the previous exercise, prove this first for $x \in \mathbb{N}$ (by induction), then for $x \in \mathbb{Z}$, and finally for $x \in \mathbb{Q}$.

Exercise 3. Let $a, b \in \mathbb{Z}$. Prove that $b - a$ divides $b^n - a^n$ for all $n \in \mathbb{N}$.

Exercise 4. Prove by induction that $7^n - 2 \cdot 4^n + 1$ is divisible by 18 for all $n \in \mathbb{N}$. Hint: Observe that $7^{n+1} - 2 \cdot 4^{n+1} + 1 = 4(7^n - 2 \cdot 4^n + 1) + 3(7^n - 1)$ for all $n \in \mathbb{N}$. You may need to do a separate proof by induction in your induction step.

Some examples from calculus

Exercises 5-10 assume familiarity with the basic techniques of differential and integral calculus.

Exercise 5. Prove that $\frac{d^n}{dx^n}(xe^x) = (n + x)e^x$ for all $n \in \mathbb{N}$.

Exercise 6. Find and prove an expression for $\frac{d^n}{dx^n}(x^2e^x)$ for all $n \in \mathbb{N}$.

Exercise 7. Let f and g be differentiable functions. Prove that

$$\frac{d^n}{dx^n}(f(x)g(x)) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x)$$

for all $n \in \mathbb{N}$, where the notation $h^{(r)}$ denotes the r^{th} derivative of h.

Exercise 8. Find and prove an expression for the n^{th} derivative of $\log(x)$ for all $n \in \mathbb{N}$.

Exercise 9. Prove that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for all $\theta \in \mathbb{R}$ and all $n \in \mathbb{N}$. Hint: You will need to use the angle addition formulae

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \quad \text{and} \quad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Exercise 10. (a) Prove that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{n+2}(x)\ dx = \frac{n+1}{n+2}\int_0^{\frac{\pi}{2}} \sin^n(x)\ dx$ for all $n \in \mathbb{N}$.

(b) Use induction to prove that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n}(x)\ dx = \frac{\pi}{2^{2n+1}}\binom{2n}{n}$ for all $n \in \mathbb{N}$.

(c) Find and prove an expression for $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n+1}(x)\ dx$ for all $n \in \mathbb{N}$.

Hint: For (a) use integration by parts; you do not need induction. For parts (b) and (c), use part (a).