Math 303 Homework 8

October 17, 2022

Recursive definitions

In Exercises 1-5, use the recursive definitions of addition, multiplication and exponentiation directly to prove the desired equation.

Exercise 1. $1 + 3 = 4$

Exercise 2. $0 + 5 = 5$

Exercise 3. $2 \cdot 3 = 6$

Exercise 4. $0 \cdot 5 = 0$

Exercise 5. $2^3 = 8$

Exercise 6. Give a recursive definition of new quantifiers $\exists^{=n}$ for $n \in \mathbb{N}$, where given a set X and a predicate $p(x)$, the logical formula $\exists^{=n} x \in X,\ p(x)$ means 'there are exactly n elements $x \in X$ such that $p(x)$ is true'. That is, define $\exists^{=0}$, and then define $\exists^{=n+1}$ in terms of $\exists^{=n}$. Hint: For example, if there are exactly 3 elements of X making $p(x)$ true, then that means that there is some $a \in X$ such that $p(a)$ is true, and there are exactly two elements $x \in X$ *other than* a making $p(x)$ true.

Exercise 7. Use the recursive definition of binomial coefficients (Definition 5.1.15) to prove directly that $\binom{4}{2} = 6$.

Exercise 8. (a) Find the number of trailing 0s in the decimal expansion of 41!.

(b) Find the number of trailing 0s in the binary expansion of 41!.

Hint: The number of trailing zeros in the base-b expansion of a natural number n is the greatest natural number r such that b^r divides n. How many times does 10 go into 41!? How many times does 2 go into 41!?

Exercise 9. Let N be a set, let $z \in N$ and let $s : N \to N$. Prove that (N, z, s) is a notion of natural numbers (in the sense of **??**) if and only if, for every set X, every element $a \in X$ and every function $f : X \to X$, there is a unique function $h : N \to X$ such that $h(z) = a$ and $h \circ f = s \circ h$.

Proofs by induction

Exercise 10. Find all natural numbers n such that $n^5 < 5^n$.

Exercise 11. Prove that $(1+x)^{123\,456\,789} \geqslant 1 + 123\,456\,789\,x$ for all real $x \geqslant -1$. Hint: Proving this by induction on x only demonstrates that it is true for *integers* $x \geqslant -1$, not *real numbers* $x \geqslant -1$. Try proving a more general fact by induction on a different variable, and deducing this as a special case. (Do you *really* think there is anything special about the natural number 123 456 789?)

Exercise 12. Let $a \in \mathbb{N}$ and assume that the last digit in the decimal expansion of a is 6. Prove that the last digit in the decimal expansion of a^n is 6 for all $n \geqslant 1$.

Exercise 13. Let $a, b \in \mathbb{R}$, and let $a_0, a_1, a_2, \ldots$ be a sequence such that $a_0 = a$, $a_1 = b$ and $a_n = \dfrac{a_{n-1} + a_{n+1}}{2}$ for all $n \geqslant 1$. Prove that $a_n = a + (b-a)n$ for all $n \in \mathbb{N}$.