Math 303 Homework 7

Images and preimages

In Exercises 1-4, find the image $f[U]$ of the subset U of the domain of the function f described in the question.

Exercise 1. $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \sqrt{1+x^2}$ for all $x \in \mathbb{R}$; $U = \mathbb{R}$.

Exercise 2. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$; $f(a, b) = a + 2b$ for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$; $U = \{1\} \times \mathbb{Z}$.

Exercise 3. $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$; $f(0) = \varnothing$ and $f(n+1) = f(n) \cup \{n\}$ for all $n \in \mathbb{N}$; $U = \mathbb{N}$.

Exercise 4. $f : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ (where $\mathbb{R}^{\mathbb{R}}$ is the set of all functions $\mathbb{R} \to \mathbb{R}$); $f(h)(x) = h(|x|)$ for all $h \in \mathbb{R}^{\mathbb{R}}$ and all $x \in \mathbb{R}$; $U = \mathbb{R}^{\mathbb{R}}$. Hint: Begin by observing that each $h \in f[\mathbb{R}^{\mathbb{R}}]$ is an even function, in the sense of **??**.

In Exercises 5-7, find the preimage $f^{-1}[V]$ of the subset V of the codomain of the function f described in the question.

Exercise 5. $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \sqrt{1+x^2}$ for all $x \in \mathbb{R}$; $V = (-5, 5]$.

Exercise 6. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$; $f(a, b) = a + 2b$ for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$; $V = \{n \in \mathbb{Z} \mid n \text{ is odd}\}$.

Exercise 7. $f : \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$; $f(A, B) = A \cap B$ for all $(A, B) \in \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$; $V = \{\varnothing\}$.

Exercise 8. Let $f : X \to Y$ be a function. For each of the following statements, either prove it is true or find a counterexample.

(a) $U \subseteq f^{-1}[f[U]]$ for all $U \subseteq X$;

(b) $f^{-1}[f[U]] \subseteq U$ for all $U \subseteq X$;

(c) $V \subseteq f[f^{-1}[V]]$ for all $V \subseteq Y$;

(d) $f[f^{-1}[V]] \subseteq V$ for all $V \subseteq Y$.

Exercise 9. Let $f : X \to Y$ be a function, let A be a set, and let $p : X \to A$ and $i : A \to Y$ be functions such that the following conditions hold:

(i) i is injective;

(ii) $i \circ p = f$; and

(iii) If $q : X \to B$ and $j : B \to Y$ are functions such that j is injective and $j \circ q = f$, then there is a unique function $u : A \to B$ such that $j \circ u = i$.

Prove that there is a unique bijection $v : A \to f[X]$ such that $i(a) = v(a)$ for all $a \in f[X]$. Hint: This problem is very fiddly. First prove that conditions (i)–(iii) are satisfied when $A = f[X]$ and p and i are chosen appropriately. Then condition (iii) in each case (for A and for $f[X]$) defines functions $v : A \to f[X]$ and $w : f[X] \to A$, and gives uniqueness of v. You can prove that these functions are mutually inverse using the 'uniqueness' part of condition (iii).

Exercise 10. Let $f : X \to Y$ be a function and let $U, V \subseteq Y$. Prove that:

(a) $f^{-1}[U \cap V] = f^{-1}[U] \cap f^{-1}[V]$;

(b) $f^{-1}[U \cup V] = f^{-1}[U] \cup f^{-1}[V]$; and

(c) $f^{-1}[Y \backslash U] = X \backslash f^{-1}[U]$.

Thus preimages preserve the basic set operations.

Exercise 11. Let $f : X \to Y$ and $g : Y \to Z$ be functions.

(a) Prove that $(g \circ f)[U] = g[f[U]]$ for all $U \subseteq X$;

(b) Prove that $(g \circ f)^{-1}[W] = f^{-1}[g^{-1}[W]]$ for all $W \subseteq Z$.

Injections, surjections and bijections

Exercise 12. (a) Prove that, for all functions $f : X \to Y$ and $g : Y \to Z$, if $g \circ f$ is bijective, then f is injective and g is surjective.

(b) Find an example of a function $f : X \to Y$ and a function $g : Y \to Z$ such that $g \circ f$ is bijective, f is not surjective and g is not injective.

Hint: Avoid the temptation to prove either part of this question by contradiction. For (a), a short proof is available directly from the definitions of 'injection' and 'surjection'. For (b), find as simple a counterexample as you can.

Exercise 13. For each of the following pairs (U, V) of subsets of $\mathbb{R}$, determine whether the specification '$f(x) = x^2 - 4x + 7$ for all $x \in U$' defines a function $f : U \to V$ and, if it does, determine whether f is injective and whether f is surjective.

(a) $U = \mathbb{R}$ and $V = \mathbb{R}$;

(b) $U = (1, 4)$ and $V = [3, 7)$;

(c) $U = [3, 4)$ and $V = [4, 7)$;

(d) $U = (3, 4]$ and $V = [4, 7)$;

(e) $U = [2, \infty)$ and $V = [3, \infty)$;

(f) $U = [2, \infty)$ and $V = \mathbb{R}$.

Exercise 14. For each of the following pairs of sets X and Y, find (with proof) a bijection $f : X \to Y$.

(a) $X = \mathbb{Z}$ and $Y = \mathbb{N}$;

(b) $X = \mathbb{R}$ and $Y = (-1, 1)$;

(c) $X = [0, 1]$ and $Y = (0, 1)$;

(d) $X = [a, b]$ and $Y = (c, d)$, where $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$.

Exercise 15. Prove that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(a, b) = \binom{a+b+1}{2} + b$ for all $(a, b) \in \mathbb{N} \times \mathbb{N}$ is a bijection. Hint: Start by proving that $\binom{m}{2} < \binom{m+1}{2}$ for all $m \geqslant 1$. Deduce that, for all $n \in \mathbb{N}$, there is a unique natural number k such that $\binom{k+1}{2} \leqslant n < \binom{k+2}{2}$. Can you see what this has to do with the function f?

Exercise 16. Let $e : X \to X$ be a function such that $e \circ e = e$. Prove that there exist a set Y and functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = e$ and $f \circ g = \mathrm{id}_Y$. Hint: Consider the set of fixed points of e—that is, elements $x \in X$ such that $e(x) = x$.