Math 303 Homework 6

Exercise 1. For each of the following equations, determine whether there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x, y \in \mathbb{R}$, the equation holds if and only if $y = f(x)$.

(a) $x + y = 1$

(b) $x^2 + y^2 = 1$

(c) $x = 0$

(d) $y = 0$

(e) $(1 + x^2)y = 1$

(f) $(1 - x^2)y = 0$

Exercise 2. Let X be a set. Prove that

$$\forall a \in X, \exists! U \in \mathcal{P}(X), (a \in U \wedge \exists! x \in X, x \in U)$$

Give an explicit description of the function $X \to \mathcal{P}(X)$ that is suggested by this logical formula.

Exercise 3. Show that there is only one function whose codomain is empty. What is its domain? Hint: Given a function $f : X \to \varnothing$, we must have $f(a) \in \varnothing$ for each $a \in X$.

Definition 4. A function $f : \mathbb{R} \to \mathbb{R}$ is **even** if $f(-x) = f(x)$ for all $x \in \mathbb{R}$, and it is **odd** if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Exercise 5. Let $n \in \mathbb{N}$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$ for all $x \in \mathbb{R}$ is even if and only if n is even, and odd if and only if n is odd.

Exercise 6. Prove that there is a unique function $f : \mathbb{R} \to \mathbb{R}$ that is both even and odd.

Exercise 7. Let $U \subseteq \mathbb{R}$, and let $\chi_U : \mathbb{R} \to \{0, 1\}$ be the indicator function of U.

(a) Prove that χ_U is an even function if and only if $U = \{-u \mid u \in U\}$;

(b) Prove that χ_U is an odd function if and only if $\mathbb{R} \backslash U = \{-u \mid u \in U\}$.

Exercise 8. Prove that for every function $f : \mathbb{R} \to \mathbb{R}$, there is a unique even function $g : \mathbb{R} \to \mathbb{R}$ and a unique odd function $h : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x) + h(x)$ for all $x \in \mathbb{R}$. Hint: Consider $f(x) + f(-x)$ and $f(x) - f(-x)$ for $x \in \mathbb{R}$.

Exercise 9. Let $\{\theta_n : [n] \to [n] \mid n \in \mathbb{N}\}$ be a family of functions such that $f \circ \theta_m = \theta_n \circ f$ for all $f : [m] \to [n]$. Prove that $\theta_n = \mathrm{id}_{[n]}$ for all $n \in \mathbb{N}$. Hint: Fix $n \in \mathbb{N}$ and consider how θ_n interacts with functions $f : [1] \to [n]$.

Exercise 10. Let X be a set and let $U, V \subseteq X$. Describe the indicator function $\chi_{U \triangle V}$ of the symmetric difference of U and V (see Definition 2.E.1) in terms of χ_U and χ_V.

Exercise 11. [This question assumes some familiarity with integral calculus.] A lie that is commonly told to calculus students is that

$$\int \frac{1}{x}\, dx = \log(|x|) + c$$

where log denotes the natural logarithm function and c is an arbitrary real constant. Prove that it is actually the case that

$$\int \frac{1}{x}\, dx = \log(|x|) + a\chi_U(x) + b\chi_V(x)$$

where a and b are arbitrary real constants, and U and V are particular (inhabited) subsets of $\mathbb{R}\setminus\{0\}$ that you should determine.