Math 303 Homework 5

Exercises 2-7 concern the *symmetric difference* of sets, defined below.

Definition 1. The **symmetric difference** of sets X and Y is the set $X \triangle Y$ defined by

$$X \triangle Y = \{a \mid a \in X \text{ or } a \in Y \text{ but not both}\}$$

Exercise 2. Prove that $X \triangle Y = (X \backslash Y) \cup (Y \backslash X) = (X \cup Y) \backslash (X \cap Y)$ for all sets X and Y.

Exercise 3. Let X be a set. Prove that $X \triangle X = \varnothing$ and $X \triangle \varnothing = X$.

Exercise 4. Let X and Y be sets. Prove that $X = Y$ if and only if $X \triangle Y = \varnothing$.

Exercise 5. Prove that sets X and Y are disjoint if and only if $X \triangle Y = X \cup Y$.

Exercise 6. Prove that $X \triangle (Y \triangle Z) = (X \triangle Y) \triangle Z$ for all sets X, Y and Z. Hint: The temptation is to write a long string of equations, but it is far less painful to prove this by double containment, splitting into cases where needed. An even less painful approach is to make a cunning use of truth tables.

Exercise 7. Prove that $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$ for all sets X, Y and Z. Hint: The hint for the previous exercise applies here too.

Definition 8. A subset $U \subseteq \mathbb{R}$ is **open** if, for all $a \in U$, there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq U$.

In Exercises 9-12 you will prove some elementary facts about open subsets of $\mathbb{R}$.

Exercise 9. For each of the following subsets of $\mathbb{R}$, determine (with proof) whether it is open:

(a) $\varnothing$;

(b) $(0, 1)$;

(c) $(0, 1]$;

(d) $\mathbb{Z}$;

(e) $\mathbb{R} \backslash \mathbb{Z}$;

(f) $\mathbb{Q}$.

Exercise 10. Prove that a subset $U \subseteq \mathbb{R}$ is open if and only if, for all $a \in U$, there exist $u, v \in \mathbb{R}$ such that $u < a < v$ and $(u, v) \subseteq U$.

Exercise 11. In this question you will prove that the intersection of finitely many open sets is open, but the intersection of infinitely many open sets might not be open.

(a) Let $n \geqslant 1$ and suppose $U_1, U_2, \ldots, U_n$ are open subsets of $\mathbb{R}$. Prove that the intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ is open.

(b) Prove that $(0, 1 + \frac{1}{n})$ is open for all $n \geqslant 1$, but that $\bigcap_{n \geqslant 1}(0, 1 + \frac{1}{n})$ is not open.

Exercise 12. Prove that a subset $U \subseteq \mathbb{R}$ is open if and only if it can be expressed as a union of open intervals—more precisely, $U \subseteq \mathbb{R}$ is open if and only if, for some indexing set I, there exist real numbers a_i, b_i for each $i \in I$, such that $U = \bigcup_{i \in I}(a_i, b_i)$.

Exercise 13. Let $\{A_n \mid n \in \mathbb{N}\}$ and $\{B_n \mid n \in \mathbb{N}\}$ be families of sets such that, for all $i \in \mathbb{N}$, there exists some $j \geqslant i$ such that $B_j \subseteq A_i$. Prove that $\bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} B_n$.