Math 303 Homework 3

August 11, 2022

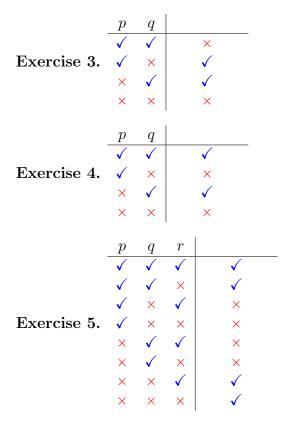
Exercise 1. Prove that

$$p \Leftrightarrow q \equiv (p \Rightarrow q) \land ((\neg p) \Rightarrow (\neg q))$$

How might this logical equivalence help you to prove statements of the form 'p if and only if q'?

Exercise 2. Prove using truth tables that $p \Rightarrow q \neq q \Rightarrow p$. Give an example of propositions p and q such that $p \Rightarrow q$ is true but $q \Rightarrow p$ is false.

In Exercises 3-6, find a logical formula whose column in a truth table is as shown.



	p	a	r	
	$\frac{P}{\checkmark}$	4 ✓	✓	
	<i>√</i>	, ,	×	×
	\checkmark	×	\checkmark	×
Exercise 6.	\checkmark	×	×	×
	×	\checkmark	\checkmark	\checkmark
	×	\checkmark	×	×
	×	×	\checkmark	\checkmark
	×	X	×	×

Exercise 7. A new logical operator \uparrow is defined by the following rules:

- (i) If a contradiction can be derived from the assumption that p is true, then $p \uparrow q$ is true;
- (ii) If a contradiction can be derived from the assumption that q is true, then $p \uparrow q$ is true;
- (iii) If r is any proposition, and if $p \uparrow q$, p and q are all true, then r is true.

This question explores this curious new logical operator.

- (a) Prove that $p \uparrow p \equiv \neg p$, and deduce that $((p \uparrow p) \uparrow (p \uparrow p)) \equiv p$.
- (b) Prove that $p \lor q \equiv (p \uparrow p) \uparrow (q \uparrow q)$ and $p \land q \equiv (p \uparrow q) \uparrow (p \uparrow q)$.
- (c) Find a propositional formula using only the logical operator \uparrow that is equivalent to $p \Rightarrow q$.

Exercise 8. Let X be \mathbb{Z} or \mathbb{Q} , and define a logical formula p by:

$$\forall x \in X, \exists y \in X, (x < y \land [\forall z \in X, \neg (x < z \land z < y)])$$

Write out $\neg p$ as a maximally negated logical formula. Prove that p is true when $X = \mathbb{Z}$, and p is false when $X = \mathbb{Q}$.

Exercise 9. Use the definition of \exists ! to write out a maximally negated logical formula that is equivalent to $\neg \exists ! x \in X$, p(x). Describe the strategy that this equivalence suggests for proving that there is not a unique $x \in X$ such that p(x) is true, and use this strategy to prove that, for all $a \in \mathbb{R}$, if $a \neq -1$ then there is not a unique $x \in \mathbb{R}$ such that $x^4 - 2ax^2 + a^2 - 1 = 0$.

Exercise 10. Define a new quantifier \forall ! such that de Morgan's laws for quantifiers (Theorem 1.3.28) hold with \forall and \exists replaced by \forall ! and \exists !, respectively.