Math 303 Homework 3

August 11, 2022

Exercise 1. Prove that

$$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge ((\neg p) \Rightarrow (\neg q))$$

How might this logical equivalence help you to prove statements of the form 'p if and only if q'?

Exercise 2. Prove using truth tables that $p \Rightarrow q \not\equiv q \Rightarrow p$. Give an example of propositions p and q such that $p \Rightarrow q$ is true but $q \Rightarrow p$ is false.

In Exercises 3-6, find a logical formula whose column in a truth table is as shown.

Exercise 3.

p	q	
✓	✓	×
✓	×	✓
×	✓	✓
×	×	×

Exercise 4.

p	q	
✓	✓	✓
✓	×	×
×	✓	✓
×	×	×

Exercise 5.

p	q	r	
✓	✓	✓	✓
✓	✓	×	✓
✓	×	✓	×
✓	×	×	×
×	✓	✓	×
×	✓	×	×
×	×	✓	✓
×	×	×	✓

Exercise 6.

p	q	r	
✓	✓	✓	✓
✓	✓	×	×
✓	×	✓	×
✓	×	×	×
×	✓	✓	✓
×	✓	×	×
×	×	✓	✓
×	×	×	×

Exercise 7. A new logical operator $\uparrow$ is defined by the following rules:

(i) If a contradiction can be derived from the assumption that p is true, then $p \uparrow q$ is true;

(ii) If a contradiction can be derived from the assumption that q is true, then $p \uparrow q$ is true;

(iii) If r is any proposition, and if $p \uparrow q$, p and q are all true, then r is true.

This question explores this curious new logical operator.

(a) Prove that $p \uparrow p \equiv \neg p$, and deduce that $((p \uparrow p) \uparrow (p \uparrow p)) \equiv p$.

(b) Prove that $p \vee q \equiv (p \uparrow p) \uparrow (q \uparrow q)$ and $p \wedge q \equiv (p \uparrow q) \uparrow (p \uparrow q)$.

(c) Find a propositional formula using only the logical operator $\uparrow$ that is equivalent to $p \Rightarrow q$.

Exercise 8. Let X be $\mathbb{Z}$ or $\mathbb{Q}$, and define a logical formula p by:

$$\forall x \in X,\ \exists y \in X,\ (x < y \wedge [\forall z \in X,\ \neg(x < z \wedge z < y)])$$

Write out $\neg p$ as a maximally negated logical formula. Prove that p is true when $X = \mathbb{Z}$, and p is false when $X = \mathbb{Q}$.

Exercise 9. Use the definition of $\exists!$ to write out a maximally negated logical formula that is equivalent to $\neg\exists! x \in X,\ p(x)$. Describe the strategy that this equivalence suggests for proving that there is not a unique $x \in X$ such that $p(x)$ is true, and use this strategy to prove that, for all $a \in \mathbb{R}$, if $a \neq -1$ then there is not a unique $x \in \mathbb{R}$ such that $x^4 - 2ax^2 + a^2 - 1 = 0$.

Exercise 10. Define a new quantifier $\forall!$ such that de Morgan's laws for quantifiers (Theorem 1.3.28) hold with $\forall$ and $\exists$ replaced by $\forall!$ and $\exists!$, respectively.