Math 303 Homework 2

August 11, 2022

Exercise 1. For fixed $n \in \mathbb{N}$, let p represent the proposition 'n is even', let q represent the proposition 'n is prime' and let r represent the proposition '$n = 2$'. For each of the following propositional formulae, translate it into plain English and determine whether it is true for all $n \in \mathbb{N}$, true for some values of n and false for some values of n, or false for all $n \in \mathbb{N}$.

(a) $(p \wedge q) \Rightarrow r$

(b) $q \wedge (\neg r) \Rightarrow (\neg p)$

(c) $((\neg p) \vee (\neg q)) \vee (\neg r)$

(d) $(p \wedge q) \wedge (\neg r)$

Exercise 2. For each of the following plain English statements, translate it into a symbolic propositional formula. The propositional variables in your formulae should represent the simplest propositions that they can.

(a) Guinea pigs are quiet, but they're loud when they're hungry.

(b) It doesn't matter that 2 is even, it's still a prime number.

(c) $\sqrt{2}$ can't be an integer because it is an irrational number.

Exercise 3. Let p and q be propositions, and assume that $p \Rightarrow (\neg q)$ is true and that $(\neg q) \Rightarrow p$ is false. Which of the following are true, and which are false?

(a) q being false is necessary for p to be true.

(b) q being false is sufficient for p to be true.

(c) p being true is necessary for q to be false.

(d) p being true is sufficient for p to be false.

In Exercises 4-7, use the definitions of the logical operators in Section 1.1 to describe what steps should be followed in order to prove the propositional formula in the question; the letters p, q, r and s are propositional variables.

Exercise 4. $(p \wedge q) \Rightarrow (\neg r)$

Exercise 5. $(p \vee q) \Rightarrow (r \Rightarrow s)$

Exercise 6. $(p \Rightarrow q) \Leftrightarrow (\neg p \Rightarrow \neg q)$

Exercise 7. $(p \wedge (\neg q)) \vee (q \wedge (\neg p))$

Exercise 8. Find a statement in plain English, involving no variables at all, that is equivalent to the logical formula $\forall a \in \mathbb{Q}, \forall b \in \mathbb{Q}, (a < b \Rightarrow \exists c \in \mathbb{R}, [a < c < b \ \wedge \ \neg(c \in \mathbb{Q})])$. Then prove this statement, using the structure of the logical formula as a guide.

Exercise 9. Find a purely symbolic logical formula that is equivalent to the following statement, and then prove it: "*No matter which integer you may choose, there will be an integer greater than it.*"

Exercise 10. Let X be a set and let $p(x)$ be a predicate. Find a logical formula representing the statement 'there are exactly two elements $x \in X$ such that $p(x)$ is true'. Use the structure of this logical formula to describe how a proof should be structured, and use this structure to prove that there are exactly two real numbers x such that $x^2 = 1$.