Math 303 Homework 1

Exercises from *Infinite Descent*.

Exercise 1 (Optional)**.** The video-sharing website *YouTube* assigns to each video a unique identifier, which is a string of 11 characters from the set

$$\{\texttt{A}, \texttt{B}, \ldots, \texttt{Z}, \texttt{a}, \texttt{b}, \ldots, \texttt{z}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}, \texttt{-}, \texttt{\_}\}$$

This string is actually a natural number expressed in base-64, where the characters in the above set represent the numbers 0 through 63, in the same order—thus `C` represents 2, `c` represents 28, `3` represents 55, and `_` represents 63. According to this schema, find the natural number whose base-64 expansion is `dQw4w9WgXcQ`, and find the base-64 expansion of the natural number 7 159 047 702 620 056 984.

Exercise 2. Let $a, b, c, d \in \mathbb{Z}$. Under what conditions is $(a + b\sqrt{2})(c + d\sqrt{2})$ an integer?

Exercise 3. Suppose an integer m leaves a remainder of i when divided by 3, and an integer n leaves a remainder of j when divided by 3, where $0 \leqslant i, j < 3$. Prove that $m + n$ and $i + j$ leave the same remainder when divided by 3.

Exercise 4. What are the possible remainders of n^2 when divided by 3, where $n \in \mathbb{Z}$?

Definition 5. A set X is **closed** under an operation $\odot$ if, whenever a and b are elements of X, $a \odot b$ is also an element of X.

In Exercises 6–12, determine which of the number sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are closed under the operation $\odot$ defined in the question.

Exercise 6. $a \odot b = a + b$

Exercise 7. $a \odot b = a - b$

Exercise 8. $a \odot b = (a - b)(a + b)$

Exercise 9. $a \odot b = (a - 1)(b - 1) + 2(a + b)$

Exercise 10. $a \odot b = \dfrac{a}{b^2 + 1}$

Exercise 11. $a \odot b = \dfrac{a}{\sqrt{b^2 + 1}}$

Exercise 12. $a \odot b = \begin{cases} a^b & \text{if } b > 0,\ b \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$

Definition 13. A complex number α is **algebraic** if $p(\alpha) = 0$ for some nonzero polynomial $p(x)$ over $\mathbb{Q}$.

Exercise 14. Let α be a rational number. Prove that α is an algebraic number.

Exercise 15. Prove that $\sqrt{2}$ is an algebraic number.

Exercise 16. Prove that $\sqrt{2} + \sqrt{3}$ is an algebraic number.

Exercise 17. Prove that $x + yi$ is an algebraic number, where x and y are any two rational numbers.